Quantum Mechanics 101

10. 2025

90% of QM $\longrightarrow$ Fourier transform.

$$\psi(x) = \langle x | \psi \rangle$$

state projected to $|x\rangle$-space

$\langle x | p \rangle$ is an eigenfunction for $\hat{p}$

$$\langle x | \hat{p} | p \rangle = p \langle x | p \rangle .$$

$$\hat{p} \longleftrightarrow -i \frac{\partial}{\partial x}$$

$$-i \frac{\partial}{\partial x} \langle x | p \rangle = p \langle x | p \rangle .$$

$$\Rightarrow \quad \langle x | p \rangle = e^{ipx}$$

$$\langle x | \psi \rangle = \int \frac{dp}{2\pi} \langle x | p \rangle \langle p | \psi \rangle$$

Fourier transform

$$= \int \frac{dp}{2\pi} e^{ipx} \underline{\tilde{\psi}(p)} .$$

state projected to $|p\rangle$-space

$$\langle p | \psi \rangle = \int dx \langle p | x \rangle \langle x | \psi \rangle$$

$$= \int dx \; e^{-ipx} \psi(x)$$

$\hat{p} \leftrightarrow -i\frac{\partial}{\partial x}$ is to realize.

$[\hat{x}, \hat{p}] = i$ in $|x\rangle$ - space

$$\langle x|\hat{p}|\psi\rangle = \int\frac{dp'}{2\pi}\, e^{ip'x}\, p' \,\langle p'|\psi\rangle$$

$$= \int\frac{dp'}{2\pi}\, -i\frac{\partial}{\partial x}\left[e^{ip'x}\langle p'|\psi\rangle\right]$$

$$= -i\frac{\partial}{\partial x}\underbrace{\int\frac{dp'}{2\pi}\, e^{ip'x}\langle p'|\psi\rangle}_{\langle x|\psi\rangle}$$

$$\hat{p} \leftrightarrow -i\frac{\partial}{\partial x}$$

$$[\hat{x}, \hat{p}] \leftrightarrow x\left(-i\frac{\partial}{\partial x}\right)\mathbb{I} - \left(-i\frac{\partial}{\partial x}\right)x\,\mathbb{I}$$

$$= i\,\mathbb{I}$$ as needed!

convention:

$\langle x|x'\rangle = \delta_{x-x'}$

$\langle p|p'\rangle = 2\pi\,\delta_{p-p'}$

in $|p\rangle$ - space

$\hat{x} \leftrightarrow i\frac{\partial}{\partial p}$ $\quad\hat{p} \leftrightarrow p.$

(Nothing special about x)

In QM : t is special (2)

quantum state evolves via.

$$i\partial_t |\psi(t)\rangle = \hat{H} |\psi(t)\rangle.$$ the Schrödinger equation

$$\rightarrow |\psi(t)\rangle = e^{-i\hat{H}t} |\psi(t=0)\rangle.$$

$\Longrightarrow$ useful to consider the eigenstate of $\hat{H}$: $\{ |n\rangle \}$

$$\hat{H}|n\rangle = E_n|n\rangle. \quad \text{(discrete } n\text{)}.$$

$|n\rangle$ satisfies

$$\hat{H}|n\rangle = E_n|n\rangle. \quad \text{TISE}$$

$$\varphi_n(x) = \langle x|n\rangle.$$

$$\langle x|\hat{H}|n\rangle = E_n \langle x|n\rangle.$$

$$\Updownarrow$$

$$\left\{ \frac{1}{2m} \left(-i\frac{d}{dx}\right)^2 + V(x) \right\} \varphi_n(x) = E_n \varphi_n(x).$$

the usual starting point

$$I \to \sum_n |n\rangle\langle n|$$

$$\langle x|\psi(t)\rangle = \psi(t,x)$$

$$= \langle x|\, e^{-i\hat{H}t}\, |\psi(t=0)\rangle$$

$$= \sum_n e^{-iE_nt}\, \varphi_n(x)\, \underline{\langle n|\psi(t=0)\rangle}$$

c_n

coefficient

eigen-mode

evolves separately at fixed $E_n \to$ frequencies

off-limit:

Energy-eigenstate $\neq$ E-space $\hookrightarrow$ FT of t

$\hookrightarrow$ Resolvent techniques

Retarded GFs.

$$\frac{1}{E-\hat{H}+i\delta}$$

come back when you know more ...

Unitary Operations

Motion $\longrightarrow$ generated by a unitary operator $\mathcal{U}_k$.

$$|x\rangle \xrightarrow{\mathcal{U}_k} |x+\Delta x\rangle.$$

$$\mathcal{U}_k = e^{-i\Delta x \hat{k}}$$

s.t

$$\mathcal{U}_k|x\rangle = e^{-i\Delta x\hat{k}}|x\rangle = |x+\Delta x\rangle$$

$\hat{k} \longrightarrow$ real, called the generator

Action of $\hat{k}$ can be clarified by

$$\langle x|\mathcal{U}_k|\psi\rangle = (\mathcal{U}_k^{-1}|x\rangle)^\dagger |\psi\rangle.$$

$$= \langle \mathcal{U}_k^{-1}x|\psi\rangle = \langle x-\Delta x|\psi\rangle$$

$$= \psi(x-\Delta x).$$

if $\Delta x \to 0$

$$\langle x | \mathcal{U}_k | \psi \rangle \approx \langle x | \left(\mathbb{I} - i \Delta x \hat{k} \right) | \psi \rangle$$

$$= \psi(x) - i \Delta x \langle x | \hat{k} | \psi \rangle$$

$$\psi(x - \Delta x) \approx \psi(x) - \Delta x \frac{\partial}{\partial x} \psi(x)$$

$$\Rightarrow \quad \langle x | \hat{k} | \psi \rangle = -i \frac{\partial}{\partial x} \psi(x).$$

$$\hat{k} \Longleftrightarrow \hat{p} \quad !$$

2 POVs

$\langle x | \mathcal{U}_k | \psi \rangle$ or $\langle \mathcal{U}_k^{-1} x | \psi \rangle$

active

basis state evolved backward.

Heisenberg Picture

$$\mathcal{U}_t |x\rangle = |x+\Delta x\rangle$$

$$\hat{x}\,\mathcal{U}_t |x\rangle = (x+\Delta x)\,|x+\Delta x\rangle$$

$$= \mathcal{U}_t\,(\hat{x}+\Delta x)\,|x\rangle$$

note $\mathcal{U}_t|x\rangle \rightarrow |x+\Delta x\rangle$

$$\Rightarrow \quad \mathcal{U}_t^{-1}\,\hat{x}\,\mathcal{U}_t = (\hat{x}+\Delta x)$$

as an operator relation

$$\mathcal{U}_t = e^{-i\Delta x\,\hat{p}}$$

$$\mathcal{U}_t^{-1} = \mathcal{U}_t^{\dagger} = e^{i\Delta x\,\hat{p}}$$

$\hat{p}$ has to be real.

$$\mathcal{U}_t^{-1}\,\hat{x}\,\mathcal{U}_t \simeq \hat{x} + i\Delta x\,[\hat{p},\hat{x}] + \cdots$$

$$= \hat{x} + \Delta x$$

$$\text{if } [\hat{x},\hat{p}] = i$$

the commutation relation is **derived**!

$$U_x = e^{-i\Delta x\hat{p}}$$

$$\hat{p} := i\frac{\partial}{\partial\Delta x}U_x\Big|_{\Delta x \to 0}$$ formally

$$\langle x|\hat{p}|\psi\rangle \leftrightarrow -i\frac{\partial}{\partial x}\langle x|\psi\rangle$$

$$U_x^{-1}\hat{x}U_x = \hat{x}+\Delta x$$

$$\swarrow$$

$$U^{-1}\hat{O}U = \hat{O}'$$

$\Rightarrow$ operator evolution

$$U_t \to e^{-i\hat{H}t}$$

$$|\psi(t)\rangle \to U_t|\psi(0)\rangle$$

$$\langle\psi(t)|\hat{O}|\psi(t)\rangle = \langle\psi(0)|\underbrace{U_t^{-1}\hat{O}U_t}_{\updownarrow\ \hat{O}_H(t)}|\psi(0)\rangle$$

$\uparrow$

Schrödinger pic.

Schrödinger pic

$|\psi_{(0)}\rangle \xrightarrow{U_t} |\psi_{(t)}\rangle$ $\quad$ $\hat{op}$ won't change.

Heisenberg pic

$|\psi_{(0)}\rangle$ unchanged but

$$\hat{O}(t) = U_t^{-1}\, \hat{O}(t=0)\, U_t$$

& $\hat{O}$ at various t
will NOT commute

$\leftrightarrow$ QM effect

Rotation :

$$D = e^{-i\alpha L^3}$$

$$L^3 = x p_y - y p_x .$$

$$D^{-1} x D = \cos\alpha \; x - \sin\alpha \; y$$
$$D^{-1} y D = \cos\alpha \; y + \sin\alpha \; x .$$

$$R = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$D^{-1} x^i D = R^{ij} x^j$$

this <u>defines</u> a vector; works also for

$$D^{-1} p^i D = R^{ij} p^j$$

Generally: for a vector operator V^k

$$[L^3, V^j] = i \, \varepsilon_{3jk} V^k$$

$$\downarrow$$

$$[L^i, V^j] = i \, \varepsilon_{ijk} V^k$$

$$[L^i, L^j] = i \, \varepsilon_{ijk} L^k \quad \longleftrightarrow \quad \vec{L} \text{ is a vector}$$

the concept of unitary $\hat{op}$ works in a general Hilbert Space:

SU(2) rotations

$$D_{\hat{n}} = e^{-i (\hat{n} \cdot \frac{1}{2} \vec{\sigma}) \theta}$$

$$D^{-1} \vec{\sigma} D = \cos\theta \, \vec{\sigma} + (1 - \cos\theta) \, \hat{n} \, \hat{n} \cdot \vec{\sigma} + \sin\theta \, (\hat{n} \times \vec{\sigma})$$

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10.2025

①

Time dependent perturbation theory
&
Fermi Golden Rule

Sakurai
P. 341

$$H \longrightarrow H_0 + V(t)$$

$$H_0 |n\rangle = E_n |n\rangle$$

$$|n(t)\rangle = e^{-iE_n t} |n\rangle$$

$\{|n(t)\rangle\}$ spans the Hilbert Space at time t

$$|\psi(t)\rangle = \sum_n |n(t)\rangle \underbrace{\langle n(t)|\psi(t)\rangle}_{c_n(t)}$$

$$= \sum_n c_n(t)\, e^{-iE_n t} |n\rangle$$

The Schrödinger Equation will take the form:

$$\sum_n (\cancel{E_n} c_n + i\dot{c}_n)\, e^{-iE_n t} |n\rangle =$$

$$\sum_n (\cancel{E_n} + V)\, c_n e^{-iE_n t}$$

$$\Rightarrow \quad \dot{c}_n = -i \sum_m e^{iE_{nm} t}\, V_{nm}\, c_m$$

$$E_{nm} = E_n - E_m$$

$$V_{nm} = \langle n | V | m \rangle$$

in matrix form $c_n \mapsto \vec{c}$

$$\dot{\vec{c}} = -i \overleftrightarrow{V}_I \cdot \vec{c}$$

$$V_I = e^{iH_0 t} V e^{-iH_0 t}$$

$$(V_I)_{nm} = e^{iE_{nm}t} V_{nm}$$

—— interaction picture naturally emerges !

How to solve such eq$\underline{n}$. ?

$$\dot{\vec{c}} = -i \overleftrightarrow{V}_I \cdot \vec{c}$$

$$\overset{?}{\Rightarrow} \quad \vec{c} = e^{-i\int^t dt' \overleftrightarrow{V}_I(t')} \vec{c}_0$$

but $[V_I(t), V_I(t')] \neq 0$.

we need to arrange in time : time-ordered

$$\vec{c} \mapsto T\, e^{-i\int^t dt' \overleftrightarrow{V}_I(t')} \vec{c}_0$$

Still need to make sense of what time order is :

Back to

$$\dot{c}_n = -i \, V_{I_{nm}} \, c_m$$

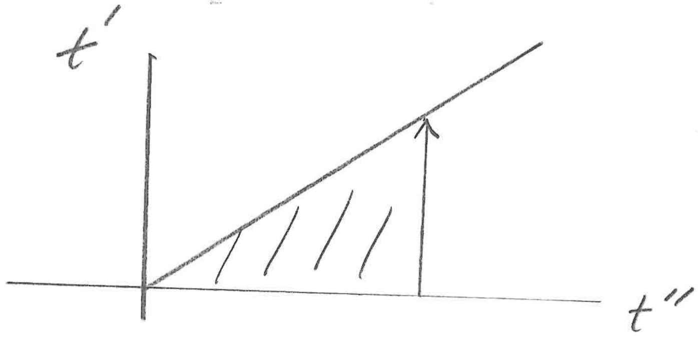
$$\Rightarrow \quad c_n(t) = c_n(0) - i \int^t dt' \, (V_I)_{nm}(t') \, \underbrace{c_m(t')}_{\text{full solution}}$$

$\rightarrow$ we can iterate order by order

Fix the initial condition

$c_n(0) = \delta_{n0}$ i.e. start from the ground state

$$c_n(t) = \delta_{n0} - i \int^t dt' \, (V_I)_{n0} + (-i)^2 \int^t dt' \, (V_I)_{nm}(t') \int^{t'} dt'' \, (V_I)_{m0}(t'') + \ldots$$



$$\int^{t} dt' \, V_I(t') \int^{t'} dt'' \, V_I(t'')$$

$$\longrightarrow \frac{1}{2!} \int^{t} dt' \int^{t} dt'' \; T\{ V_I(t'), V_I(t'') \}$$

$\frac{1}{N!}$ generally

time order

$$T\{ A(t_1), A(t_2) \} = \begin{cases} A(t_1)\, A(t_2) & \text{if } t_1 > t_2 \\ A(t_2)\, A(t_1) & \text{if } t_2 > t_1 \end{cases}$$

$$T\{ e^{-i\int^{t} dt' \, V_I(t')} \} \leftrightarrow$$

$$\sum_{n} \frac{(-i)^n}{n!} \, T\{ \int^{t} dt_1' \, V_I(t_1') \int^{t} dt_2' \, V_I(t_2') \cdots \}$$

$$//$$

The Fermi Golden Rule can be obtained already from leading order correction

$$c_n(t) \approx \delta_{n0} - i \int^t dt' \, (V_I)_{n0}$$

<u>level 1</u>

$$V(t) = \theta(t) \, \bar{V}$$

if final state $n \neq 0$

$$c_{n\neq 0}(t) \approx -i \int_0^t dt' \, \bar{V}_{n0} \, e^{iE_{n0}t'}$$

$$= -i \, \bar{V}_{n0} \, \frac{(e^{iE_{n0}t} - 1)}{iE_{n0}}$$

probability can be defined

$$|c_{n\neq 0}|^2 = |\bar{V}_{n0}|^2 \frac{1}{E_{n0}^2} |e^{iE_{n0}t} - 1|^2$$

$$= |\bar{V}_{n0}|^2 \frac{1}{E_{n0}^2} \, 4 \sin^2 \frac{E_{n0}t}{2}$$

$$\xrightarrow{t \to \infty} |\bar{V}_{n0}|^2 \, 2\pi \, \delta_{E_n - E_0} \, t$$

prove later

Fermi Golden Rule :

$$\Gamma_{i\to f} = \overset{\text{transition}}{\text{rate}} = \frac{d}{dt}|c_f|^2$$

$$= |\bar{V}_{fi}|^2 \, 2\pi \, \delta_{E_f - E_i}$$

$$\Gamma = \sum_f \Gamma_{i\to f}$$

$$= \sum_f |\bar{V}_{fi}|^2 \, 2\pi \, \delta_{E_f - E_i}$$

$$\longrightarrow \int dE_f \, D(E_f) \, |\bar{V}_{fi}|^2 \, 2\pi \, \delta_{E_f - E_i}$$

↳ DOS

for a continuum of final states

Now we go back to the proof

method # 1

use the identity

Sakurai P.343.

$$\lim_{\alpha\to\infty} \frac{\sin^2 \alpha x}{\alpha x^2} = \pi \, \delta(x)$$

$$\alpha \to t/2 .$$

method # 2
(PML).

$$|C_{n\neq 0}|^2 = |\bar{V}_{n0}|^2 \int_0^t dt' \int_0^t dt'' \; e^{iE_{n0}(t'-t'')}$$

change the variables to
$\tau = t' - t''$
$s = t''$

$$\longrightarrow |\bar{V}_{n0}|^2 \int_0^t ds \int_{-s}^{t-s} d\tau \; e^{iE_{n0}\tau}$$

$$\longrightarrow |\bar{V}_{n0}|^2 \int_{-t}^{t} d\tau \; \underline{\int_{R(s)} ds} \; e^{iE_{n0}\tau}$$

$\hookrightarrow t - |\tau|$

$t \to \infty$

$$\longrightarrow |\bar{V}_{n0}|^2 \; 2\pi \delta_{E_n - E_0} \times t$$

↑ keep only ~~the first~~ term

the 2nd term can be worked out:

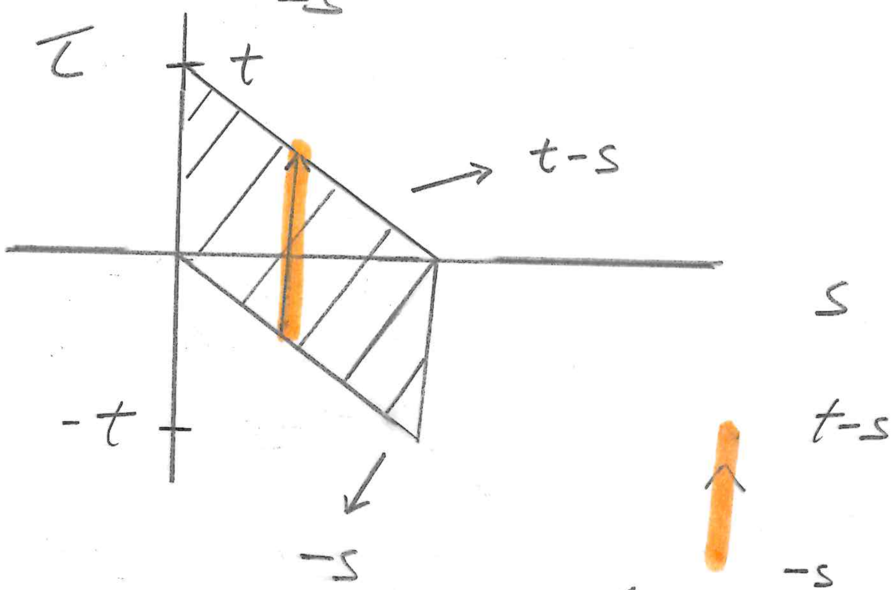
$$\longrightarrow |\bar{V}_{n0}|^2 \frac{2}{E_{n0}^2}$$

but can be neglected at large t

this is quite a useful trick we record it here

$$\int_0^t dt_1 \int_0^t dt_2 \; f(t_1 - t_2)$$

$$= \int_0^t ds \int_{-s}^{t-s} d\tau \; f(\tau)$$



for a given s

how to integrate the same region but for fixed τ

$$\left.\begin{array}{ll} \text{if} & \tau > 0 \\ s: & 0 \to t-\tau \\ \text{if} & \tau < 0 \\ s: & -\tau \to t \end{array}\right\}$$

this defines the range of s : $R(s)$

$$\int_{R(s)} ds = \begin{array}{ll} t-\tau & \text{if } \tau > 0 \\ t+\tau & \phantom{\text{if }} \tau < 0 \end{array}$$

$$\to \quad t - |\tau| \quad //$$

level 2 expert derivation

$$V(t) \rightarrow e^{\eta t}\,\bar{V} \qquad t \rightarrow -\infty$$

$V \rightarrow 0$

adiabatic switching-on

$$C_f^{(1)} \rightarrow -i\,\bar{V}_{fi} \int_{-\infty}^{t} dt'\; e^{iE_{fi}t'}\, e^{\eta t'}$$

$$= \frac{\bar{V}_{fi}}{E_i - E_f + i\eta}\; e^{i(E_{fi} - i\eta)t}$$

$$|C_f^{(1)}|^2 \rightarrow |\bar{V}_{fi}|^2 \frac{e^{2\eta t}}{E_{if}^2 + \eta^2}$$

$$T_{i\to f} = \frac{d}{dt} |C_f^{(1)}|^2 \quad (\text{for large } t)$$

$$= |\bar{V}_{fi}|^2 \frac{2\eta}{E_{if}^2 + \eta^2}\; e^{2\eta t}$$

$$\xrightarrow{\eta \to 0} |\bar{V}_{fi}|^2\, 2\pi\, \delta_{E_i - E_f}$$

to see this

$$\frac{1}{x+i\delta} = P\frac{1}{x} - i\pi\,\delta(x).$$

$$\mathrm{Im}\,\frac{1}{x+i\delta} = \frac{-\delta}{x^2+\delta^2} \longleftrightarrow -\pi\,\delta(x).$$

$$\Rightarrow \frac{2\delta}{x^2+\delta^2} \longleftrightarrow 2\pi\,\delta(x).$$

as $\delta \to 0$

$\Rightarrow$ the same Golden Rule is derived w/o trouble

2nd order effects :

$$c_f^{(0)} = \delta_{fi}$$

$$c_f^{(1)} = \bar{V}_{fi} \frac{1}{E_{if} + i\eta} e^{i(E_{fi} - i\eta)t}$$

$$c_f^{(2)} = \sum_{n'} \int_{-\infty}^{t} dt' \, e^{i(E_{fn'} - i\eta)t'} (-i \bar{V}_{fn'}) \times$$

$$\int_{-\infty}^{t} dt'' \, e^{i(E_{n'i} - i\eta)t''} (-i \bar{V}_{n'i}).$$

$$= \sum_{n'} \int_{-\infty}^{t} dt' \, e^{i(E_{fi} - 2i\eta)t'} \frac{-i \bar{V}_{fn'} \bar{V}_{n'i}}{E_{in'} + i\eta}$$

$$= \sum_{n'} \frac{\bar{V}_{fn'} \bar{V}_{n'i}}{[E_{if} - 2i\eta][E_{in'} + i\eta]} e^{i(E_{fi} - 2i\eta)t}$$

if $f \to i$

$$c_{f\to i} \approx 1 - i\,\bar{V}_{ii}\,\frac{1}{\eta}\,e^{\eta t} +$$

$$(-i)^2\,|\bar{V}_{ii}|^2\,\frac{1}{2\eta^2}\,e^{2\eta t} +$$

$$-i\sum_{n'}{}' |\bar{V}_{in'}|^2\,\frac{1}{E_{in'}+i\eta}\,\frac{1}{2\eta}\,e^{2\eta t}$$

$\sum_{n'}{}'$ means $n' \neq i$

as $\eta \to 0$ $\quad c_{f\to i}$ diverges !

Instead we consider

$$\dot{c}_{f\to i} \approx -i\,\bar{V}_{ii}\,e^{\eta t} +$$

$$(-i)^2\,|\bar{V}_{ii}|^2\,\frac{1}{\eta}\,e^{2\eta t} +$$

$$-i\sum_{n'}{}' |\bar{V}_{in'}|^2\,\frac{1}{E_{in'}+i\eta}\,e^{2\eta t}$$

still diverges as $\eta \to 0$

but when we construct

$$\frac{\dot{c}_{f\to i}}{c_{f\to i}} \simeq \frac{\left\{ -i\bar{V}_{ii}\, e^{\eta t} + (-i)^2 |\bar{V}_{ii}|^2 \frac{1}{\eta} e^{2\eta t} + (-i) \sum_{n'}' |\bar{V}_{in'}|^2 \frac{1}{E_{in'}+i\eta} e^{2\eta t} + \dots \right\}}{1 - i\bar{V}_{ii} \frac{1}{\eta} e^{\eta t} + \dots}$$

$$\simeq -i\bar{V}_{ii} \quad -i \sum_{n'}' |\bar{V}_{in'}|^2 \frac{1}{E_{in'}+i\eta}$$

safe to take $\eta \to 0$

$\forall$ div terms cancel.

$\frac{d}{dt} \ln c_{f\to i}$ is finite

$$c^{R}_{f\to i} \propto e^{-i\Delta t}$$

$$\Delta = \bar{V}_{ii} + \underbrace{\sum_{n'}' |\bar{V}_{in'}|^2 \frac{1}{E_i - E_{n'} + i\eta}}_{\text{complex}}$$

$$= \Delta_R + i\Delta_I$$

$$\Delta_R = \underline{\text{energy shift}}$$

$$= \bar{V}_{ii} + \sum_{n'}{}' |V_{in'}|^2 \, P \frac{1}{E_i - E_{n'}}$$

$$\Delta_I = -\pi \sum_{n'}{}' |V_{in'}|^2 \, \delta_{E_i - E_{n'}}$$

$$\Rightarrow -2\Delta_I = \sum_{n'}{}' |V_{in'}|^2 \, 2\pi \, \delta_{E_i - E_{n'}}$$

$$= \sum_{n'}{}' \Gamma_{i \to n'} = \Gamma \quad /\!/ \ \underline{\text{width}}$$

as dictated by Fermi Golden Rule

$$|C^R_{f \to i}|^2 = e^{2\Delta_I t} \approx 1 + 2\Delta_I t$$

$\updownarrow$

prob. of $i \to f \approx i$

$$\sum_{f = n' \neq i}{}' \Gamma_{i \to n'} t \quad \leftrightarrow \quad \text{prob. of } i \to f = n' \neq i$$

$$|C^R_{f \to i}|^2 + \sum_{n'}{}' \Gamma_{i \to n'} t = 1$$

$\Rightarrow$ unitarity $\leftrightarrow$ conservation of prob.

Further extension:

a) Beyond 2nd order

$$\dot{c}_f \approx -i \, \bar{V}_{fi} \, e^{iE_{fi}t} +$$

$$-i \sum_{n'} \frac{\bar{V}_{fn'} \bar{V}_{n'i}}{E_i - E_{n'} + i\eta} e^{iE_{fi}t} + \dots$$

$$= -i \left\{ \bar{V}_{fi} + \sum_{n'} \frac{\bar{V}_{fn'} \bar{V}_{n'i}}{E_i - E_{n'} + i\eta} + \dots \right\} e^{iE_{fi}t}$$

$$\longrightarrow -i \, T_{fi} \, e^{iE_{fi}t}$$

T is T-matrix

in Lippmann-Schwinger eqn.

$$T = V + V \underbrace{\frac{1}{E - \hat{H}_0 + i\delta}}_{\hat{G}_0} T$$

$\hat{G}_0$ the resolvent

b) harmonic perturbation

in cases we have studied so far as $t \to \infty$ only $E_f \to E_i$ contributes.

$\to$ if V has a characteristic fequency ω

$$V(t) = V e^{i\omega t} + V^{+} e^{-i\omega t}$$

$$C_f^{(1)} \to V_{fi} \frac{1}{E_i - \omega - E_f + i\eta} e^{i(E_{fi} + \omega - i\eta)t} +$$

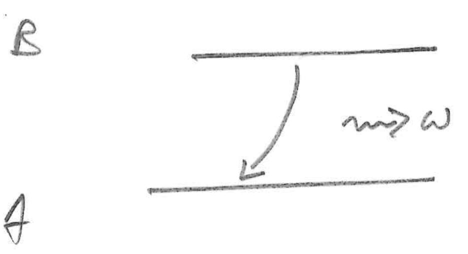
$$V^{+}_{fi} \frac{1}{E_i + \omega - E_f + i\eta} e^{i(E_{fi} - \omega - i\eta)t}$$

1st term $E_f \to E_i - \omega$ emission.
2nd term $E_f \to E_i + \omega$ absorption.

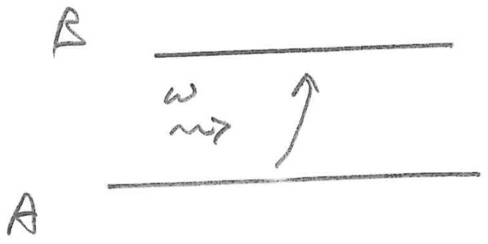
$$|C_f^{(1)}|^2 \simeq \begin{cases} |V_{fi}|^2 \, 2\pi \, \delta(E_f - E_i + \omega) \, t \\ |V^{+}_{fi}|^2 \, 2\pi \, \delta(E_f - E_i - \omega) \, t \end{cases}$$

so we release / extract E to the external source

Emission



Absorption.



$$E_A = E_B - \omega$$

$$E_B = E_A + \omega.$$

$$\langle \underset{f}{A} | V | \underset{i}{B} \rangle$$

$$\langle \overset{f}{B} | V^\dagger | \overset{i}{A} \rangle = \langle A|V|B \rangle^*$$

the strengths ~~same~~ the same

$$|V_{AB}|^2 = |V^\dagger_{BA}|^2$$

⇙ as a consequence

$$\frac{\Gamma^{\text{emit}}_{B \to A+\omega}}{D_A} = \frac{\Gamma^{\text{absorb.}}_{A+\omega \to B}}{D_B}.$$

└ DoS of A

that's detailed balance!