UV & IR Div in QM

A) standard perturbation theory

$$H|\psi\rangle = E|\psi\rangle$$

$$H \to H_0 + V$$

if V is small

$$|\psi\rangle \to |\psi^{(0)}\rangle + |\psi^{(1)}\rangle + |\psi^{(2)}\rangle + \cdots$$

$$E \to E^{(0)} + E^{(1)} + E^{(2)}$$

to measure the influence of V

we also need the basis $\{|n\rangle\}$
which satisfies

$$H_0|n\rangle = \varepsilon_n|n\rangle$$

Master Eqn.

$$(H_0 + V)\left(|\psi^{(0)}\rangle + |\psi^{(1)}\rangle + |\psi^{(2)}\rangle + \cdots\right) =$$

$$\left(E^{(0)} + E^{(1)} + E^{(2)} + \cdots\right)\left(|\psi^{(0)}\rangle + |\psi^{(1)}\rangle + |\psi^{(2)}\rangle + \cdots\right)$$

$\to$ extract the result order by order

0-th order is trivial.

$$H_0 |\psi^{(0)}\rangle = E^{(0)} |\psi^{(0)}\rangle$$

$\longrightarrow$ we pick $|\psi^{(0)}\rangle \longrightarrow |0\rangle$

$E^{(0)} \longrightarrow \mathcal{E}_0$

1st order

$$V|\psi^{(0)}\rangle + H_0 |\psi^{(1)}\rangle = E^{(0)}|\psi^{(1)}\rangle + E^{(1)}|\psi^{(0)}\rangle$$

projecting onto $\langle m|$

$$\langle m|V|0\rangle + \mathcal{E}_m \langle m|\psi^{(1)}\rangle = \mathcal{E}_0 \langle m|\psi^{(1)}\rangle + E^{(1)} \langle m|0\rangle.$$

Separating the 2 cases:

$m = 0$

$$\langle 0|V|0\rangle + \mathcal{E}_0 \langle 0|\psi^{(1)}\rangle = \mathcal{E}_0 \langle 0|\psi^{(1)}\rangle + E^{(1)} \langle 0|0\rangle$$

$$\Rightarrow \quad E^{(1)} = \langle 0|V|0\rangle. \quad //$$

which is easy to understand:
interaction V changes the Energy by
$\langle 0|V|0\rangle$

$m \neq 0$

$$\langle m|V|0\rangle + \varepsilon_m \langle m|\psi^{(1)}\rangle = \varepsilon_0 \langle m|\psi^{(1)}\rangle + E^{(1)} \underbrace{\cancel{\langle m|0\rangle}}_{0 \text{ by orthogonality}}.$$

$$\langle m|\psi^{(1)}\rangle = \frac{\langle m|V|0\rangle}{\varepsilon_0 - \varepsilon_m} \quad \text{for } m \neq 0$$

$\langle 0|\psi^{(1)}\rangle$ is <u>not</u> fixed <u>by</u> the master eq.

$\Rightarrow$ fix it <u>by</u> normalization

$$\langle \psi|\psi\rangle = 1$$

$$\langle 0|0\rangle + \left(\langle \psi^{(1)}|0\rangle + \langle 0|\psi^{(1)}\rangle\right) + \cdots = 1$$

0 to this order.

$$\langle 0|\psi^{(1)}\rangle = 0.$$

2nd order

$$\langle m|V|\psi^{(1)}\rangle + \langle m|H_0|\psi^{(2)}\rangle = E^{(0)}\langle m|\psi^{(2)}\rangle +$$
$$E^{(1)}\langle m|\psi^{(1)}\rangle +$$
$$E^{(2)}\langle m|\psi^{(0)}\rangle$$

$m = 0$

cancel

$$\langle 0|V|\psi^{(1)}\rangle + \mathcal{E}_0\langle 0|\psi^{(2)}\rangle = \mathcal{E}_0\langle 0|\psi^{(2)}\rangle +$$
$$E^{(1)}\langle 0|\psi^{(1)}\rangle$$
$$E^{(2)}\langle 0|0\rangle$$

$$E^{(2)} = \langle 0|\left[V - E^{(1)}\right]|\psi^{(1)}\rangle$$

$\langle 0|V|0\rangle$
measure the further shift

$$= \sum_m \underline{\langle 0|(V - E^{(1)})|m\rangle}\langle m|\psi^{(1)}\rangle$$

↳ naturally exclude $m = 0$ as

$E^{(1)} = \langle 0|V|0\rangle$

derived previously

$$\longrightarrow \sum_m{}' \langle 0|(V - E^{(1)})|m\rangle\langle m|\psi^{(1)}\rangle$$

useless $\to 0$ by orthogonality

$$E^{(2)} = \sum_m{}' \frac{\langle 0|V|m\rangle\langle m|V|0\rangle}{\varepsilon_0 - \varepsilon_m}$$

//

$m \neq 0$

$$\langle m|V|\psi^{(1)}\rangle + \varepsilon_m \langle m|\psi^{(2)}\rangle = \varepsilon_0 \langle m|\psi^{(2)}\rangle + E^{(1)}\langle m|\psi^{(1)}\rangle + \cancel{E^{(2)}\langle m|0\rangle}$$

orthogonality

$$\langle m|\psi^{(2)}\rangle = \frac{\langle m|(V - E^{(1)})|\psi^{(1)}\rangle}{\varepsilon_0 - \varepsilon_m}$$

$$= \frac{\sum_{m'} \langle m|V|m'\rangle\langle m'|\psi^{(1)}\rangle - E^{(1)}\langle m|\psi^{(1)}\rangle}{\varepsilon_0 - \varepsilon_m}$$

0 by choice

$$= \frac{\langle m|V|0\rangle\cancel{\langle 0|\psi^{(1)}\rangle} + \sum_{m'}{}' \langle m|V|m'\rangle \frac{\langle m'|V|0\rangle}{\varepsilon_0 - \varepsilon_{m'}} - E^{(1)}\langle m|\psi^{(1)}\rangle}{\varepsilon_0 - \varepsilon_m}$$

$$= \frac{1}{\varepsilon_0 - \varepsilon_m} \sum_{m'}{}' \frac{V_{mm'}V_{m'0}}{\varepsilon_0 - \varepsilon_{m'}} - \frac{V_{m0}V_{00}}{(\varepsilon_0 - \varepsilon_m)^2}$$

//

$\langle 0|\psi^{(2)}\rangle$ can again be determined by normalization

$$0 = \langle 0|\psi^{(2)}\rangle + \langle \psi^{(2)}|0\rangle + \langle \psi^{(1)}|\psi^{(1)}\rangle$$

$$\langle 0|\psi^{(2)}\rangle = -\frac{1}{2}\sum_m |\langle \psi^{(1)}|m\rangle|^2$$

$$= -\frac{1}{2}\sum_m{}' \frac{|V_{m0}|^2}{(\varepsilon_0-\varepsilon_m)^2}$$

//

B) Brillouin - Wigner Perturbation

$$(E - H)\,|\psi\rangle = 0$$

$$(E - H_0)\,|\psi\rangle = V\,|\psi\rangle$$

$$\Rightarrow \quad |\psi\rangle = |\psi_0\rangle + \frac{1}{E - H_0} V\,|\psi\rangle$$

$$(E - H_0)\,|\psi_0\rangle = 0$$

we also use the basis $\{|n\rangle\}$

$$H_0\,|n\rangle = \varepsilon_n\,|n\rangle$$

to obtain $|\psi\rangle$ we consider

$$c_n := \langle n|\psi\rangle = \langle n|\psi_0\rangle + \frac{1}{E - \varepsilon_n}\langle n|V|\psi\rangle$$

$$= \langle n|\psi_0\rangle + \sum_m \frac{1}{E - \varepsilon_n} V_{nm}\, c_m$$

$\langle n|V|m\rangle$

this is an implicit matrix eqn for c_n's

$\longrightarrow$ to be solved self-consistently we pick $|\psi_0\rangle \longrightarrow |0\rangle$

$$c_n = \delta_{n0} + \sum_m \frac{1}{E - \varepsilon_n} V_{nm} c_m$$

$$\simeq \delta_{n0} + \frac{1}{E - \varepsilon_n} V_{n0} + \sum_m \frac{1}{E - \varepsilon_n} V_{nm} \frac{1}{E - \varepsilon_m} V_{m0} + \ldots$$

w iteration.

this is <u>almost</u> an expansion in V

<u>But</u> we need to know <u>E</u>

which can be obtained via

$$\langle 0 | E - H_0 | \psi \rangle = \langle 0 | V | \psi \rangle$$

$$E = \varepsilon_0 + \frac{\langle 0 | V | \psi \rangle}{\langle 0 | \psi \rangle}$$

$$= \varepsilon_0 + \frac{1}{c_0} \sum_m V_{0m} c_m$$

$$= \varepsilon_0 + \frac{V_{00} + \sum_{m,m'} \frac{V_{0m} V_{mm'}}{E - \varepsilon_m} c_{m'}}{1 + \sum_m \frac{1}{E - \varepsilon_0} V_{0m} c_m}$$

First we obtain E up to 2nd order in V

$$E = \mathcal{E}_0 + \frac{V_{00} + \sum_{m,m'} \frac{V_{0m} V_{mm'}}{E - \mathcal{E}_m} c_{m'}}{1 + \sum_m \frac{1}{E - \mathcal{E}_0} V_{0m} c_m}$$

both side involves E

$$\simeq \mathcal{E}_0 + \frac{V_{00} + \sum_m \frac{V_{0m} V_{m0}}{E - \mathcal{E}_m}}{1 + \frac{1}{E - \mathcal{E}_0} V_{00}}$$

c_m to leading order $\to \delta_{m0}$

now we can take $E \to \mathcal{E}_0 + \delta$

keep track of IR div

$$\longrightarrow \mathcal{E}_0 + V_{00} - \cancel{V_{00}^2 \frac{1}{\delta}} + \sum_m{}' \frac{V_{0m} V_{m0}}{\mathcal{E}_0 - \mathcal{E}_m} + \cancel{\frac{V_{00}^2}{\delta}}$$

exclude $m = 0$

$$E \simeq \mathcal{E}_0 + V_{00} + \sum_m{}' \frac{V_{0m} V_{m0}}{\mathcal{E}_0 - \mathcal{E}_m} \quad //$$

E is safe for $\delta \to 0$ IR div

This is not true for c_n's.

$$C_0 \approx 1 + \frac{1}{E - \varepsilon_0} V_{00} +$$

$$\sum_m \frac{1}{E - \varepsilon_0} V_{0m} \frac{1}{E - \varepsilon_m} V_{m0}$$

take

$$E \to \varepsilon_0^+ + V_{00} + \cdots$$

$$\uparrow$$

$$\varepsilon_0 + \delta$$

$$C_0 \longrightarrow 1 + \frac{V_{00}}{\delta} - \cancel{\frac{V_{00}^2}{\delta^2}} +$$

$$\frac{1}{\delta} \sum_m{}' \frac{1}{\varepsilon_0 - \varepsilon_m} V_{0m} V_{m0} + \cancel{\frac{1}{\delta^2} V_{00}^2}$$

$$C_{n \neq 0} \approx \frac{V_{n0}}{E - \varepsilon_n} + \sum_m \frac{1}{E - \varepsilon_n} \frac{1}{E - \varepsilon_m} V_{nm} V_{m0} + \cdots$$

$$\longrightarrow \frac{V_{n0}}{\varepsilon_0 - \varepsilon_n} - \frac{V_{n0} V_{00}}{(\varepsilon_0 - \varepsilon_n)^2} + \sum_m \frac{1}{\varepsilon_0 - \varepsilon_n} \frac{1}{\varepsilon_0 - \varepsilon_m} V_{nm} V_m$$

the last term is dangerous.

at $m = 0$

$$\frac{1}{\delta} \frac{1}{\mathcal{E}_0 - \mathcal{E}_n} V_{n0} V_{00}$$

$$C_{n\neq 0} = \frac{V_{n0}}{\mathcal{E}_0 - \mathcal{E}_n} - \frac{V_{n0} V_{00}}{(\mathcal{E}_0 - \mathcal{E}_n)^2} + \sum_m{}' \frac{V_{nm} V_{m0}}{(\mathcal{E}_0 - \mathcal{E}_n)(\mathcal{E}_0 - \mathcal{E}_m)}$$

$$+ \frac{1}{\delta} \frac{1}{\mathcal{E}_0 - \mathcal{E}_n} V_{n0} V_{00}$$

Again very IR - div as $\delta \to 0$.

concider the normalization condition.

$$N^2 = \langle \psi | \psi \rangle = |C_0|^2 + \sum_n{}' |C_n|^2$$

$$= 1 + \frac{2 V_{00}}{\delta} + \frac{2}{\delta} \sum_m{}' \frac{1}{\mathcal{E}_0 - \mathcal{E}_m} V_{0m} V_{m0}$$

$$+ \frac{V_{00}^2}{\delta^2} + \sum_n{}' \frac{|V_{n0}|^2}{(\mathcal{E}_0 - \mathcal{E}_n)^2}$$

$$N \simeq 1 + \frac{V_{00}}{\delta} + \frac{1}{\delta} \sum_n{}' \frac{1}{\mathcal{E}_0 - \mathcal{E}_n} |V_{0n}|^2$$

not trivial

$$+ \frac{1}{2} \sum_n{}' \frac{|V_{n0}|^2}{(\mathcal{E}_0 - \mathcal{E}_n)^2} + \cancel{\left(\frac{1}{2} - \frac{1}{8} 2^2 \right)} \frac{V_{00}^2}{\delta^2}$$

c_n's $\sim \frac{1}{d}$

N also has $1/d$

$$\frac{c_0}{N} \approx \frac{1 + \frac{V_{00}}{d} + \frac{1}{d}\sum_m' \frac{1}{\varepsilon_0 - \varepsilon_m}|V_{0m}|^2}{\left\{1 + \frac{V_{00}}{d} + \frac{1}{d}\sum_m' \frac{|V_{0m}|^2}{\varepsilon_0 - \varepsilon_m} + \frac{1}{2}\sum_n' \frac{|V_{0n}|^2}{(\varepsilon_0 - \varepsilon_n)^2}\right\}}$$

$$\approx 1 - \frac{1}{2}\sum_n' \frac{|V_{0n}|^2}{(\varepsilon_0 - \varepsilon_n)^2}$$

$\checkmark$ IR-div terms cancel

$$\frac{c_{n\neq 0}}{N} \approx \frac{\frac{V_{0n}}{\varepsilon_0 - \varepsilon_n} - \frac{1}{(\varepsilon_0 - \varepsilon_n)^2} V_{n0} V_{00} + \sum_m' \frac{V_{nm} V_{m0}}{(\varepsilon_0 - \varepsilon_n)(\varepsilon_0 - \varepsilon_m)} + \frac{1}{d}\frac{V_{n0} V_{00}}{\varepsilon_0 - \varepsilon_n}}{1 + \frac{V_{00}}{d} + \cdots}$$

only need order V

again the IR-div cancel.

$$\frac{C_{n\neq 0}}{N} \simeq \frac{V_{0n}}{\varepsilon_0-\varepsilon_n} - \frac{V_{n0}V_{00}}{(\varepsilon_0-\varepsilon_n)^2} + \sum_m{}' \frac{V_{nm}V_{m0}}{(\varepsilon_0-\varepsilon_n)(\varepsilon_0-\varepsilon_m)}$$

Summary :

$$\frac{C_0}{N} \longrightarrow 1 - \frac{1}{2}\sum_n{}' \frac{|V_{0n}|^2}{(\varepsilon_0-\varepsilon_n)^2}$$

$$\frac{C_{n\neq 0}}{N} \longrightarrow \frac{V_{0n}}{\varepsilon_0-\varepsilon_n} - \frac{V_{n0}V_{00}}{(\varepsilon_0-\varepsilon_n)^2} + \sum_m{}' \frac{V_{nm}V_{m0}}{(\varepsilon_0-\varepsilon_n)(\varepsilon_0-\varepsilon_m)}$$

$$E = \varepsilon_0 + V_{00} + \sum_m{}' \frac{|V_{0m}|^2}{\varepsilon_0-\varepsilon_m}$$

$$|A|^2_{0\to 0} = |C_0^N|^2$$

$$\longrightarrow 1 - \sum_n{}' \frac{|V_{n0}|^2}{(\varepsilon_0 - \varepsilon_n)^2}$$

$$|\Delta A|^2_{\substack{0\to \text{other} \\ n}} = \sum_n{}' |C_n^N|^2$$

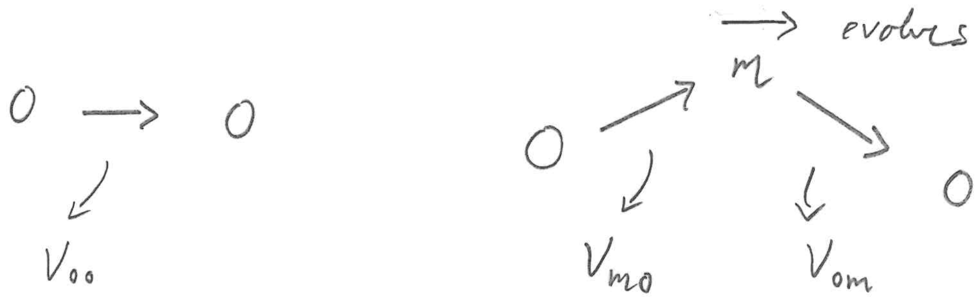
$$\simeq \sum_n{}' \frac{|V_{n0}|^2}{(\varepsilon_0 - \varepsilon_n)^2}$$

$$|A|^2_{0\to 0} + |\Delta A|^2_{0\to \text{other}} = 1 \qquad //$$

c) UV divergence

$$\Delta E = E - \mathcal{E}_0$$

$$= V_{00} + \sum_m{}' \frac{V_{0m} V_{m0}}{\mathcal{E}_0 - \mathcal{E}_m}$$



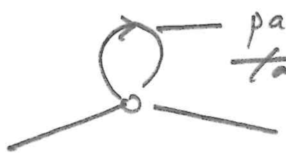
$$\frac{1}{E - \hat{H}_0 + i\delta}$$

Folklore:

① the system explores $\forall$ intermediate states w arbitrary energies $\mathcal{E}_m$ $\longrightarrow$ uncertainty principle allows the borrowing of E from ~~vacuum~~.

~ off-shell

② ~ loops in QFT



particle can take arbitrarily high E

$\hookrightarrow$ origin of UV div.

③ states $\rightarrow$ unstable when coupled to continuum let

$$V_{0m} = \langle 0|V|m\rangle \longrightarrow H_1 \quad \text{(constant)}$$

for $m \neq 0$

&

$$|m\rangle \longrightarrow |\vec{k}\rangle \qquad \mathcal{E}_m \to \frac{\vec{k}^2}{2m}.$$

$$\Delta E^{(2)} = \sum_m{}' \frac{|H_1|^2}{\mathcal{E}_0 - \mathcal{E}_m + i\delta}$$

$$\longrightarrow V \int \frac{d^3k}{(2\pi)^3} \frac{|H_1|^2}{\mathcal{E}_0 - \mathcal{E}_k + i\delta}$$

$\mathcal{E}_k \to \frac{\vec{k}^2}{2m}$

① $k \to \infty$ mode leads to UV div.

physically there will be some natural cutoff. $\Leftarrow$

$$\propto -\int d^3k \frac{1}{k^2} \sim -\Lambda_{uv}.$$

② $\mathrm{Im}\, \Delta E^{(2)}$

$$\longrightarrow -\frac{1}{2} V \int \frac{d^3k}{(2\pi)^3} |H_1|^2 \, 2\pi \, \delta_{\mathcal{E}_0 - \mathcal{E}_k}$$

coupling to continuum $\longrightarrow$ induce an imaginary part in E

$$\Delta E = \Delta_R + i\Delta_I$$

$$\Delta_R = V_{00} + \sum_m{}' \underline{P} \frac{1}{\varepsilon_0 - \varepsilon_m} |V_{om}|^2$$

$$\Delta_I = -\frac{1}{2} \sum_m{}' |V_{om}|^2 \, 2\pi \, \delta(\varepsilon_0 - \varepsilon_m)$$

$$= -\frac{1}{2} \underline{\Gamma}$$

↳ Fermi Golden Rule

↳ width ~ decay rate

$$\psi \sim e^{-i\Delta E t}$$

$$\sim e^{-i\Delta_R t} e^{\Delta_I t} \to e^{-i\Delta_R t} e^{-\frac{1}{2}\Gamma t}$$

prob. $0 \to 0$

$$|\psi|^2 \longrightarrow e^{-\Gamma t} \approx 1 - \Gamma t$$

prob. $0 \to$ other

$$\sum_m{}' |V_{om}|^2 \, 2\pi \, \delta_{\varepsilon_0 - \varepsilon_m} \, t = \Gamma t$$

unitarity : Prob. $0 \to 0$ + Prob. $0 \to$ others $= 1$

$$\Gamma := -2\,\mathrm{Im}\,\Sigma$$

$$\Delta E = V_{00} + \mathrm{Re}\,\Sigma$$

if needed

Resonances in QFT

degen., coupling to continuum in QM.

further ref.

Barry Holstein Adv. QM p. 52

Doron QM notes.