Topics

1. Symmetries of hadrons
2. Translational Invariance & Feynman Rules
3. SU(2) chiral symmetry

10. 2023

Symmetries of hadrons

$p \longleftrightarrow n$ isospin symmetry

e.g. $\mathcal{U}|n\rangle = |p\rangle \qquad \mathcal{U} = e^{-i\alpha Q}$

$$m_p = \langle p|H|p\rangle$$
$$= \langle n|\mathcal{U}^{+} H \mathcal{U}|n\rangle$$
$$\xrightarrow{\text{sym.}} \langle n|H|n\rangle = m_n$$

$\Leftrightarrow \quad \mathcal{U}^{+} H \mathcal{U} = H \quad$ or $\quad [H, Q] = 0$

also

$[H, \mathcal{U}] = 0$

① Symmetry manifests in spectrum
↳ VS SSB

② $i\partial_t \langle Q\rangle = \langle [Q, H] \rangle$

$\longrightarrow 0$ if Q is a symmetry

$\langle Q\rangle$ is a constant (in time)

i.e. conserved charge

$SU(N)$ symmetry appears a lot in hadron physics.

$SU(N_f)$ flavor: $\nearrow$ isospin

$N_f = 2$ $\quad u \leftrightarrow d$

$N_f = 3$ $\quad u, d, s$

$\neq$ strong isospin

$SU(N_c)$ color $\quad$ weak isospin

$\begin{pmatrix} R \\ G \\ B \end{pmatrix}$ $\quad \begin{pmatrix} \nu \\ e \end{pmatrix}, \begin{pmatrix} u \\ d \end{pmatrix}$ doublet

$N_c = 3$ $\quad N_c = 2$

Isospin helps to classify particles & interactions

Isospin is respected by QCD

$\Rightarrow$ $u \not\rightarrow d$ $\quad$ flavor is conserved

$c \not\rightarrow s$

but weak interactions do not respect that

$n \rightarrow p + e^- + \bar{\nu}_e$

$\left(d \rightarrow u + e^- + \bar{\nu}_e \right)$

classify the light baryon family

$I = 1/2 \quad N^*$

$I = 3/2 \quad \Delta$

$\left.\begin{array}{ll} I = 0 & \Lambda \\ I = 1 & \Sigma \end{array}\right\}$ $S = -1$ hyperon

Group theory : level 1

$SU(2)$

$\downarrow$ special, unitary — $\det U = 1$, $U^{-1} = U^\dagger$

2 → dim. of the fundamental repr.

$\{ M : SU(2) \}$

generated by generators

$\{ \tau^a \}$: 3 of them for $SU(2)$

$N_c^2 - 1$ in $SU(N_c)$ → adjoint repr.

s.t.

$$M = e^{-i\alpha^a \tau^a}$$

$$\tau^{a\dagger} = \tau^a \qquad \alpha^a\text{'s are real}$$

$$\Rightarrow M^{-1} = M^\dagger = e^{i\alpha^a \tau^a}$$

$$\ln \det M = 0 = \mathrm{tr} \ln M$$

$$= i\alpha^a \,\mathrm{tr}\, \tau^a$$

$$\Rightarrow \mathrm{tr}\, \tau^a = 0$$

τ^a has to be traceless

$$[\tau^a, \tau^b] = i\varepsilon_{abc} \tau^c \qquad \text{for } SU(2)$$

$$[T^a, T^b] = i f_{abc} T^c \qquad \text{for } SU(3)$$

$$\mathrm{tr}\, T^a T^b = \frac{1}{2} \delta^{ab} \quad \text{as normalization}$$

$\vec{\tau} = \frac{1}{2}\vec{\sigma}$ for SU(2)

3 dof $\longleftrightarrow$ π^+ π^0 π^-

for SU(3)

↳ fund $3/n$ $\begin{pmatrix} R \\ G \\ B \end{pmatrix}$

↳ adj. $A_\mu = A_\mu^a T^a$

$a = 1, 2, \dots, 8$.

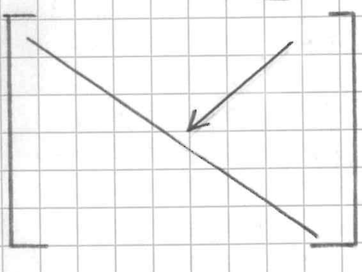
ref. Howard Georgi

Group theory : level 2

Generators of $SU(N_c)$ satisfy

$T^a = T^{a\dagger}$

how many free params ? → complex off diag.



$2\times(1+2+3+\cdots+N_c-1)$

$+ N_c - 1$ → traceless

↳ diagonal elements have to be real.

$= N_c^2 - 1$

also

$\mathrm{tr}\, T^a = 0 \Rightarrow N_c - 1$ independent diagonal elements.

↓

very special: $\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ Pauli's

λ_3, λ_8 in Gell-Mann's

Diagonal $\longrightarrow$ basis we use are eigenstates

$\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix} \Rightarrow \uparrow$ or $\downarrow$ along z - direction when we choose $\sigma_z = \begin{pmatrix}1 & \\ & -1\end{pmatrix}$

$N_c - 1$ diagonal generators

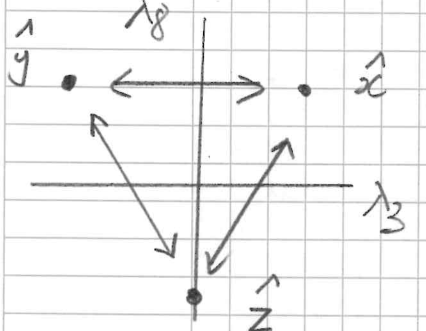
$\Longrightarrow$ cartan subgroup $\longrightarrow$ root system

$SU(2) \rightarrow 3 = 1 + 2 \times 1$

$SU(3) \rightarrow 8 = 2 + \underline{3 \times 2}$

$SU(4) \rightarrow 15 = 3 + 4 \times 3$

SU(3)



2 quantum no. (λ_3, λ_8)

$\hat{x}, \hat{y}, \hat{z}$ are represented as points

6 links move between points

the cartan elements define an $(N_c - 1)$- dim root space

In this space, N_c basis vecs $\longleftrightarrow$ N_c points

which then define $N_c(N_c - 1)$ links between these points

$\updownarrow$ "other generators"

Examples

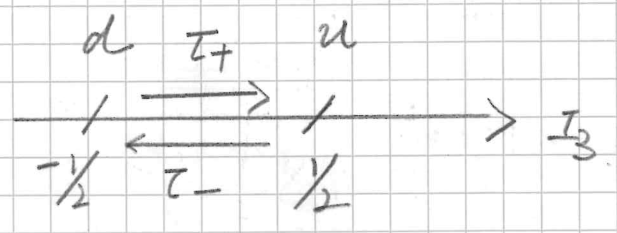
SU(2)

$$T_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\downarrow \qquad\qquad \downarrow$$

$u \qquad\qquad d \quad \rightarrow$ classified by eigenvalues ~~of~~ T_3.



$$T_+ = T_1 + i T_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} d \rightarrow u \\ u \rightarrow 0 \end{matrix}$$

$$T_- = T_1 - i T_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \begin{matrix} u \rightarrow d \\ d \rightarrow 0 \end{matrix}$$

SU(3)

$$T_3 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix} \frac{1}{2} \; ; \; T_8 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix} \frac{1}{\sqrt{3}}$$

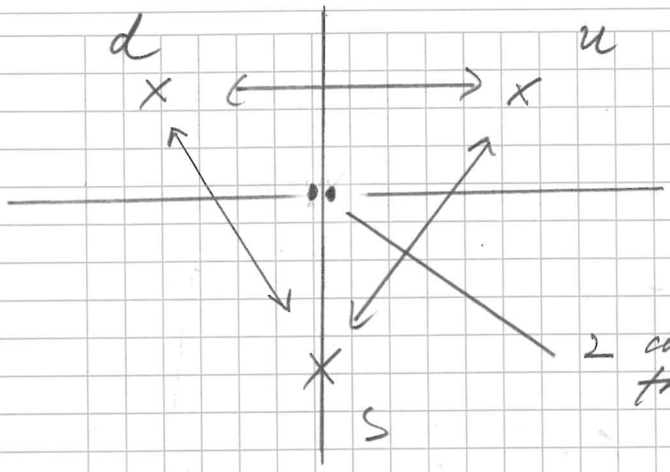
$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow$ classified according to pairs of eigenvalues (T_3, T_8)

$$u \leftrightarrow \left(1/2, \; 1/\sqrt{3}\right)$$

$$d \leftrightarrow \left(-1/2, \; 1/\sqrt{3}\right)$$

$$s \leftrightarrow \left(0, \; -2/\sqrt{3}\right).$$

also called
root vecs



6 links
to move around
basis vec (points)
weigh vecs

2 cartan elements not moving things.

Additional notes #1

$\binom{u}{d}$ SU(2) doublet

Anti-particles ?

$\binom{-\bar{d}}{\bar{u}}$ convention $SU(\bar{2})$.

why? $u \to I_z = 1/2$ $\quad \bar{u} \to I_z = -1/2$

$\bar{d} \to I_z = +1/2$

why the $-$ sign ?

if we want $\bar{u} \leftrightarrow u^*$; $\bar{d} \leftrightarrow d^*$

$q = \binom{u}{d}$ $\qquad q' = \hat{u} \binom{u}{d}$

& we naturally expect

$$\bar{q}' = \hat{u}\, \bar{q}$$

does it work ?

Not if we choose $\bar{q} = \binom{d^*}{u^*}$

e.g. $\hat{u} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\hat{u} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} -id \\ iu \end{pmatrix} = \begin{pmatrix} u' \\ d' \end{pmatrix} \quad \text{—— (1)}$$

complex conjugate

$$\hat{u}^* \begin{pmatrix} u^* \\ d^* \end{pmatrix} = \begin{pmatrix} id^* \\ -iu^* \end{pmatrix}$$

$$\Rightarrow \hat{u} \begin{pmatrix} d^* \\ u^* \end{pmatrix} = \begin{pmatrix} iu^* \\ -id^* \end{pmatrix}$$

that's <u>not</u> what we expect if we want (1)

$$\hat{u}\,\bar{q} = \bar{q}' = \begin{pmatrix} -i\,\bar{q}_2 \\ i\,\bar{q}_1 \end{pmatrix} \leftrightarrow \begin{matrix} -iu^* \\ id^* \end{matrix}$$

just as

$$\hat{u}\,q = q' = \begin{pmatrix} -iq_2 \\ iq_1 \end{pmatrix}$$

$\Rightarrow$ works if

$$\bar{q} = \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix} \rightarrow \begin{pmatrix} -d^* \\ u^* \end{pmatrix}$$

Verify this!

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} -d^* \\ u^* \end{pmatrix} = \begin{pmatrix} -i(u^*) \\ i(-d^*) \end{pmatrix}$$

consistent w the form

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} -id \\ iu \end{pmatrix}$$

$$\left.\begin{array}{l} \bar{q}_1 \to -i\,\bar{q}_2 \\ \bar{q}_2 \to i\,\bar{q}_1 \end{array}\right\} \quad \begin{array}{l} q_1 \to -i\,q_2 \\ q_2 \to i\,q_1 \end{array}$$

& the 1st line can also be derived via complex conjugation of the second

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} -u^* \\ -d^* \end{pmatrix} = \begin{pmatrix} id^* \\ -iu^* \end{pmatrix} \quad //$$

Hf. Thomson p. 221

Additional notes #2

$$M \to \{ \mathbb{1}_{N_c},\ T^a \}.$$

$$M = m_0 \mathbb{1}_c + m^a\, 2T^a$$

$$m_0 = \mathrm{tr}M / N_c$$

$$m^a = \mathrm{tr}(T^a M)$$

A useful formula for $SU(N_c)$

$$M = \frac{\mathrm{tr} M}{N_c} \mathbb{I}_{N_c} + 2\, \mathrm{tr}(M T^a)\, T^a$$

$$M_{\alpha\beta} = \frac{M_{\alpha'\alpha'}}{N_c} \delta_{\alpha\beta} + 2\, M_{\alpha'\beta'} T^a_{\beta'\alpha'} T^a_{\alpha\beta}.$$

$$M_{\alpha\beta} \to \delta_{\alpha\rho} \delta_{\beta\omega}.$$

$$\Rightarrow T^a_{\omega\rho} T^a_{\alpha\beta} = \frac{1}{2}\left(\delta_{\alpha\rho}\delta_{\beta\omega} - \frac{1}{N_c}\delta_{\rho\omega}\delta_{\alpha\beta}\right) \quad //$$

sum over $N_c^2 - 1$ a implied

Ref. Cheng & Li p. 110.

Stack them together

mesons

$SU(2)$ $\otimes$ $SU(\bar{2})$

$d \quad u$ $\otimes$ $\bar{u} \quad -\bar{d}$

$\rightarrow$ $\bar{u}d$, $\bar{u}u - \bar{d}d$, $-\bar{d}u$ $\oplus$ $\bar{u}u + \bar{d}d$

π^-, π^0, π^+; σ

$$2 \otimes \bar{2} = 3 \oplus 1$$

$SU(2) \otimes SU(2) \otimes SU(2)$ baryons

$\rightarrow$ $\left(dd,\ ud+du,\ uu \oplus ud-du \right) \otimes$

$\rightarrow$ ddd, uuu $\oplus$ partial sym. $\oplus$ partial A-sym.

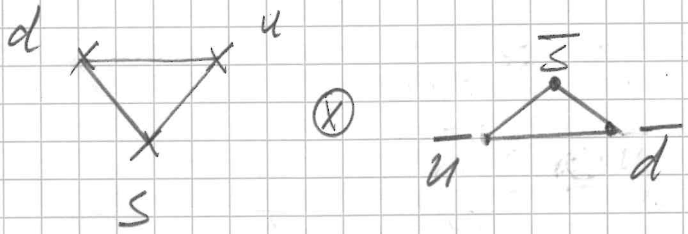
Δ^- Δ^0 Δ^+ Δ^{++}; $d(ud-du)$ $(ud-du)u$

$$2 \otimes 2 \otimes 2 = \underline{4} \oplus \underline{2 \oplus 2}$$

Δ - family

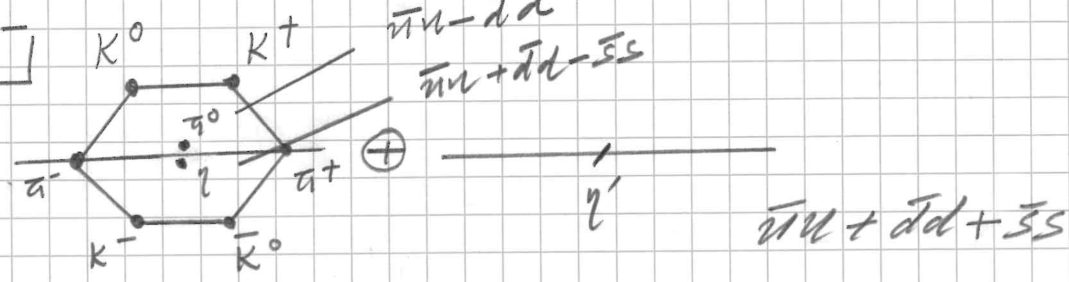
$\hookrightarrow$ recombine to form N^* - family

$SU(3) \otimes SU(\bar{3})$



$\boxed{J^P = 0^-}$

=

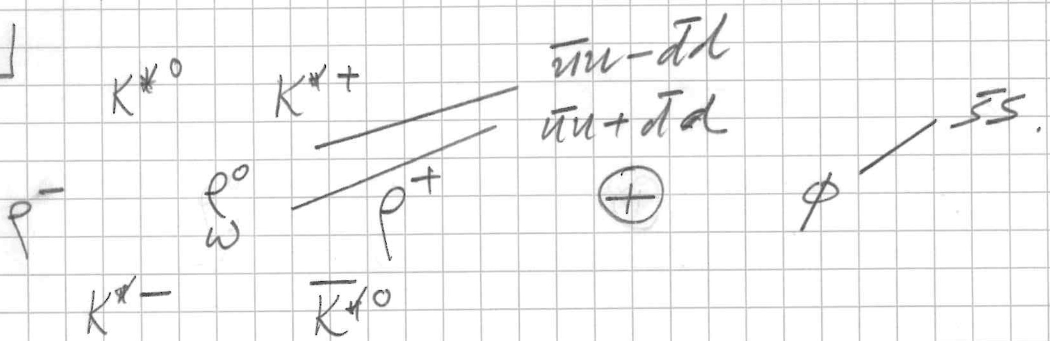


$3 \otimes \bar{3} = 8 \oplus 1$

nature demands "ideal mixing"

Also for 1^-

$\boxed{1^-}$

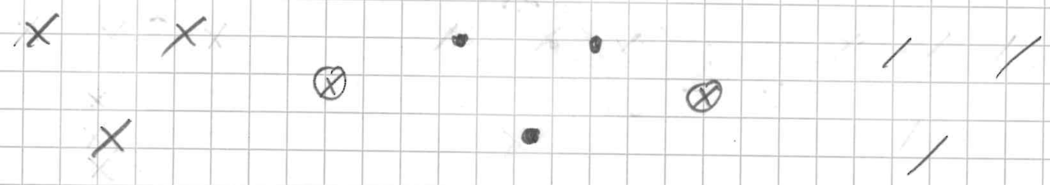


$3 \otimes 3 \otimes 3$ tough!

$= (6 \oplus \bar{3}) \otimes 3$

$= 10 \oplus 8 + 8 \oplus 1$

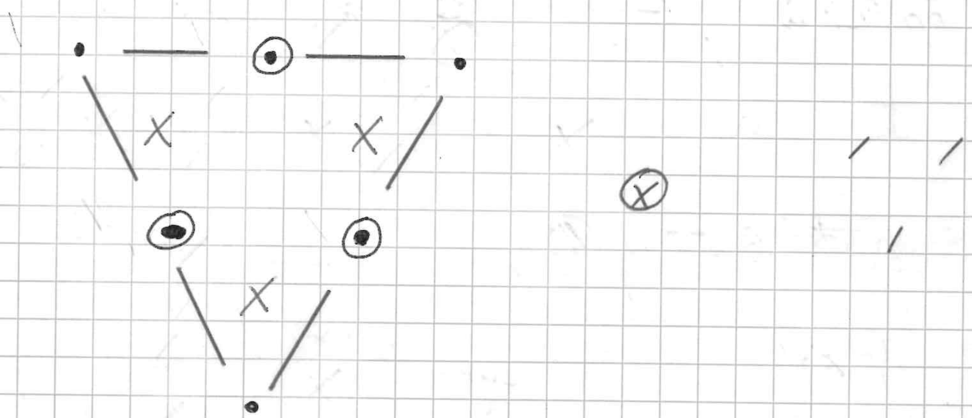
graphical



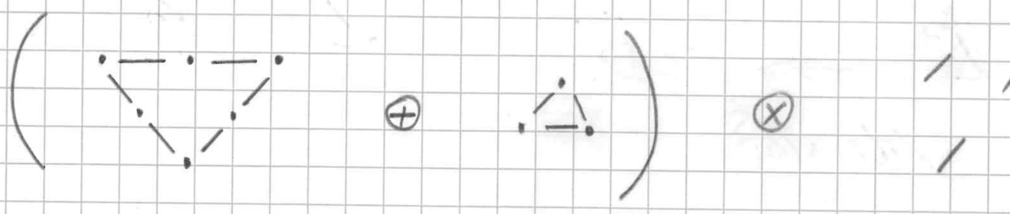
x = old

• = new

$=$



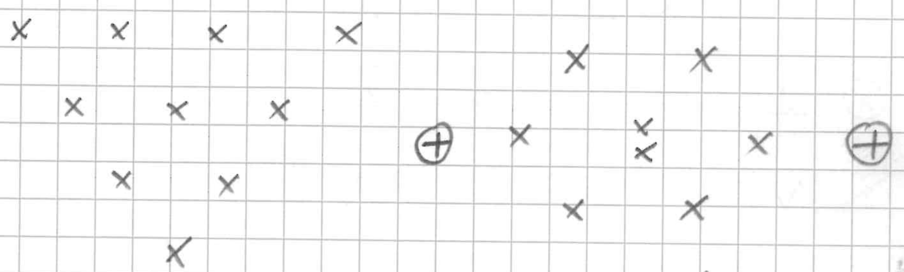
$=$



6

$\bar{3}$ (not 3)

$=$



10-state decuplet

octet

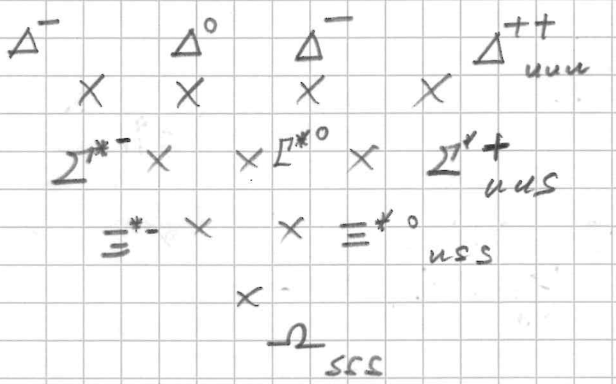
$\bar{3} \times 3$

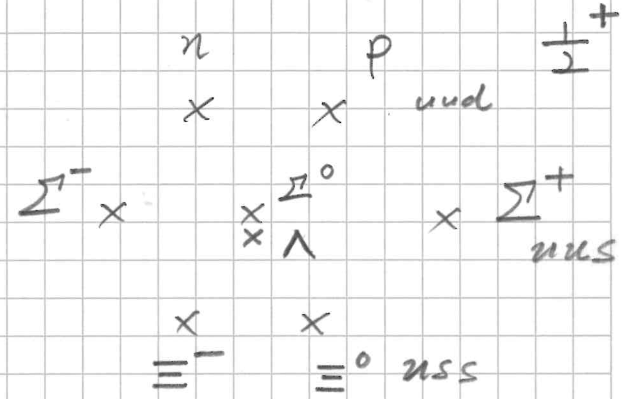
$8 \oplus 1$ already done!

$$3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3$$

$$= 10 \oplus \underline{8 \oplus 8} \oplus 1$$

decuplet $\quad \frac{3}{2}^+$ $\qquad\qquad$ remix etc.





$\forall$ satisfy : $\quad Q = I_z + \frac{1}{2}(B + S + \cdots)$ $\quad$ Gell-mann - Nishijima

2 more stories from this :

① $\Delta^{++} \sim \underline{u_\uparrow u_\uparrow u_\uparrow} \longrightarrow$ all fermionic wavefunction

$j = 3/2$ $\qquad \forall$ symmetric $\qquad \Longrightarrow$ anti-symmetric

color SU(3)

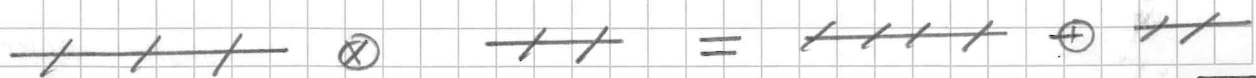
$\Delta^{++} \longrightarrow$ QCD. $\qquad \Longrightarrow$ proposed as a solution.

literally

② $\Delta \longleftrightarrow \pi N \hookrightarrow I = 1/2$, $\ \pi \hookrightarrow I = 1$ $\qquad$ qqq vs $q\bar{q}$ $qqq\bar{q}$ $\qquad$ quark model vs hadrons continuum

$$3 \otimes 2 = 4 \oplus 2$$

$\pi \quad N \qquad \Delta \quad N^*$



Implementation on theory :

① Gauge transformation

$$\psi \to e^{-ig\alpha^a T^a}\psi$$

$\to$ 3x3 matrix acting on $\psi \sim \begin{pmatrix} R \\ G \\ B \end{pmatrix}$

$$A^\mu \to A^\mu + \partial^\mu \alpha$$

conf $\leftrightarrow$ singlet only

$$A^\mu = A^\mu_a T^a$$

$\hookrightarrow$ adjoint space $\sim R\bar{G}, \cdots$

$\to$ 8 in $3 \otimes \bar{3}$

② chiral transformation

$$\psi \to e^{-i\alpha^a \tau^a \gamma_5}\psi$$

$\to$ 2x2 flavor struct.

$\to$ spinor (Dirac)

$\hookrightarrow$ flavor doublet

u, d $\to$ u: 4-component Dirac spinor

More fancy :

$$\Sigma = \sigma \mathbb{1}_2 + i\,\vec{\pi}\cdot 2\vec{\tau}$$

$$= \begin{pmatrix} \sigma + i\pi_3 & \pi_2 + i\pi_1 \\ -\pi_2 + i\pi_1 & \sigma - i\pi_3 \end{pmatrix}$$

$$\det \Sigma = \sigma^2 + \vec{a}^2$$

$\longrightarrow$ invariant under χ rotation more fancy results (to be proven later)

$$\bar{q}\,(\sigma - i\,\vec{\pi}\cdot 2\vec{t}\,\gamma_5)\,q$$ is chiral invariant

$$\updownarrow$$

$$\bar{q}_L \Sigma q_R + \bar{q}_R \Sigma^+ q_L$$

$$P_{L/R} = \frac{1}{2}\left(I \pm \gamma_5\right)$$

$$\frac{1}{2}\,\mathrm{tr}\left(\Sigma^+\Sigma\right) \longleftrightarrow \left(\sigma^2 + \vec{\pi}^2\right)$$

<u>breaking piece ?</u>

$$-\bar{q}\,\hat{m}\,q \Rightarrow \propto \mathrm{tr}\left(\hat{m}\Sigma + \Sigma^+\hat{m}^+\right)$$

inv. under V

but NOT inv. under AV