11. 2025

Green's function basics

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Simple example in Maxwell's eqn.

$$\nabla \cdot \vec{E} = \rho_{\vec{x}}$$

$$\vec{E} = -\nabla \varphi_{\vec{x}}$$

electrostatic potential $\varphi_{\vec{x}}$

$$\Rightarrow \quad -\nabla^2 \varphi_{\vec{x}} = \rho_{\vec{x}}$$

What do we mean by this?

$$\Rightarrow \quad \varphi_{\vec{x}} = \frac{-1}{\nabla^2_{\vec{x}}} \rho_{\vec{x}}$$

$$= \int d^3x' \, \underline{\frac{-1}{\nabla^2_{\vec{x}}} \delta^3_{\vec{x}-\vec{x}'}} \, \rho_{\vec{x}'}$$

GF : $G_{\vec{x}\vec{x}'}$

$G_{\vec{x}\vec{x}'}$ satisfies

$$-\nabla^2_{\vec{x}} G_{\vec{x}\vec{x}'} = \delta^3_{\vec{x}-\vec{x}'}$$

& trivially

$$-\nabla^2_{\vec{x}} \left\{ \underline{\int d^3x' \, G_{\vec{x}\vec{x}'} \, \rho_{\vec{x}'}} \right\} = \rho_{\vec{x}}$$

$\Updownarrow$ is indeed the soln : $\varphi_{\vec{x}}$

No need to find a sol^n for each $\rho_{\vec{x}}$

$\longrightarrow$ search for the solution of $G_{\vec{x}\vec{x}'}$

$$G_{\vec{x}\vec{x}'} = \frac{-1}{\nabla_x^2}\delta^3_{\vec{x}-\vec{x}'} = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \frac{1}{\vec{k}^2}$$

$$= \frac{1}{4\pi |\vec{x}-\vec{x}'|}$$

FT of $1/\vec{k}^2$

FT

$$\frac{1}{\vec{k}^2} \longleftrightarrow \frac{1}{r}$$

$$\frac{1}{\vec{k}^2+m^2} \longleftrightarrow \frac{1}{r}e^{-mr}$$

$$\nabla^2_{\vec{x}} \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta^3_{\vec{x}-\vec{x}'}$$

$$\Longleftrightarrow \quad G_{\vec{x}\vec{x}'} = \frac{-1}{\nabla^2_{\vec{x}}}\delta^3_{\vec{x}-\vec{x}'} = \frac{1}{4\pi|\vec{x}-\vec{x}'|}$$

$\Downarrow$

$$\varphi_{\vec{x}} = \int d^3x' \frac{1}{4\pi|\vec{x}-\vec{x}'|}\rho_{\vec{x}'}$$

if $\rho_{\vec{x}} = \delta^3_{\vec{x}-\vec{a}}\, Q$

$$\varphi_{\vec{x}} = \frac{Q}{4\pi|\vec{x}-\vec{a}|}$$

the Gauss law is realized as

$$\vec{E} = -\nabla \varphi_{\vec{x}} = \frac{e}{4\pi R^2}\hat{R}$$

$$e = \int d^3x\, \rho_{\vec{x}} = \int d^3x\, e\, \delta^3_{\vec{x}-\vec{a}}$$

$$= \int d\vec{A}\cdot\vec{E} = \int d^3x\, \nabla\cdot\vec{E}$$

$$= \int d^3x\, \frac{e}{4\pi}\left[-\nabla^2 \frac{1}{|\vec{x}-\vec{a}|}\right]$$

Similarly

$$(-\nabla_{\vec{x}}^2 + m^2)\, G_{\vec{x}\vec{x}'} = \delta^3_{\vec{x}\vec{x}'}$$

$$G_{\vec{x}\vec{x}'} = \frac{-1}{\nabla^2 - m^2}\delta^3_{\vec{x}\vec{x}'}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \frac{1}{\vec{k}^2 + m^2}$$

$$= \frac{1}{4\pi|\vec{x}-\vec{x}'|} e^{-m|\vec{x}-\vec{x}'|}$$

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Going further

$$\left[\partial_t^2 - \nabla_{\vec{x}}^2 + m^2\right] G_{xx'} = -\delta^4_{x-x'}$$

$$G_{xx'} = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-x')} \frac{1}{p^2 - m^2 + i\delta}$$

$$= \frac{-1}{\partial_x^2 + m^2}\, \delta^4_{x-x'}$$

$\Rightarrow$ introduce the p^0 variable

$$\frac{1}{p^{0\,2} - \vec{p}^{\,2} - m^2} = \frac{1}{p^{0\,2} - \mathcal{E}_{\vec{p}}^{\,2}}$$

QFT propagator $\rightarrow$ QFT = QM + special relativity

$p^0 = \mathcal{E}_{\vec{p}}$ is on-shell

but in QM:

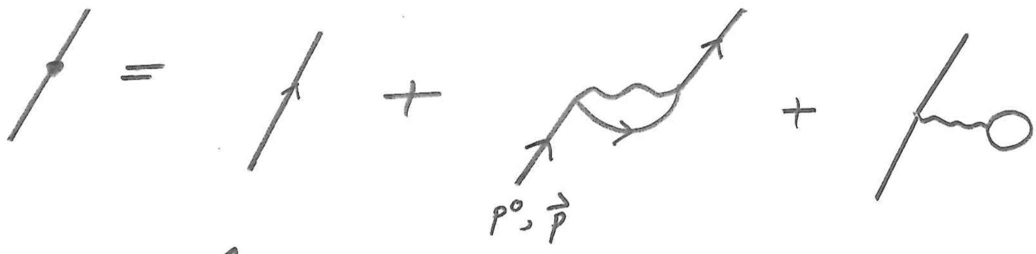
$p^0 \neq \mathcal{E}_{\vec{p}} \Leftrightarrow$ off-shell

Folklore:

$\Delta t\, \Delta E \sim 1$

allows ΔE: borrow E from vac $\Rightarrow$ Not just dictated by eqn. of motion.

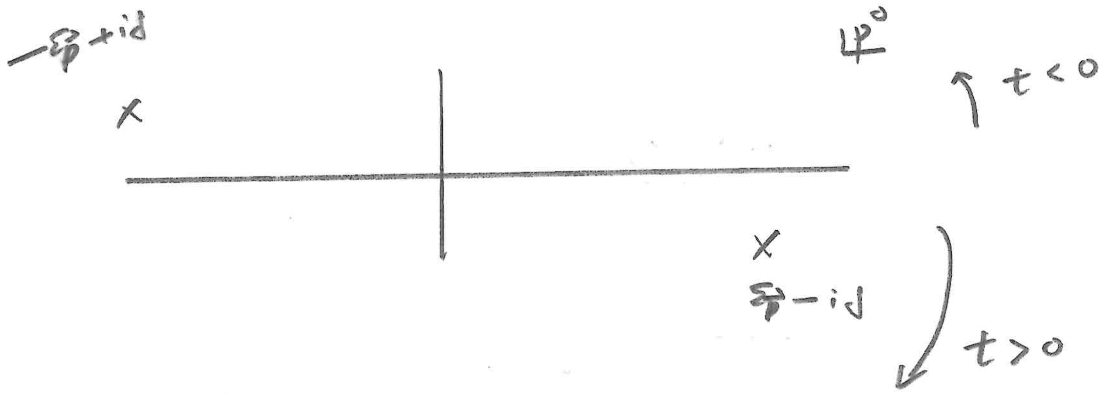
$\forall$ paths are allowed: Path Integral = quantization



$+ \cdots$

$$i G_{p^0, \vec{p}} = \frac{i}{p^2 - m^2 + i\delta}$$ propagator

$$i G_{t \vec{x}} = \int \frac{d^4p}{(2\pi)^4} e^{-i p \cdot x} \frac{i}{p^2 - m^2 + i\delta}$$



$$= \int \frac{d^3p}{(2\pi)^3} e^{-i \varepsilon_p |t|} e^{i \vec{p} \cdot \vec{x}} \frac{1}{2\varepsilon_{\vec{p}}}$$

this is "on-shell"

$$\varphi_x = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_{\vec{p}}}} \left(e^{-ip\cdot x} a_{\vec{p}} + e^{ip\cdot x} a^+_{\vec{p}} \right)$$

$$p^0 = \varepsilon_{\vec{p}}$$

$$\langle 0 | a_{\vec{p}} \, a^+_{\vec{p}} | 0 \rangle = (2\pi)^3 \delta^3_{\vec{p}-\vec{p}'}$$

$$\langle 0 | T\{ \varphi_x \varphi_y \} | 0 \rangle$$

$$= \theta(x^0 - y^0) \langle 0 | \varphi_x \varphi_y | 0 \rangle +$$
$$\theta(y^0 - x^0) \langle 0 | \varphi_y \varphi_x | 0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\varepsilon_{\vec{p}}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} e^{-i\varepsilon_{\vec{p}} |x^0 - y^0|}$$

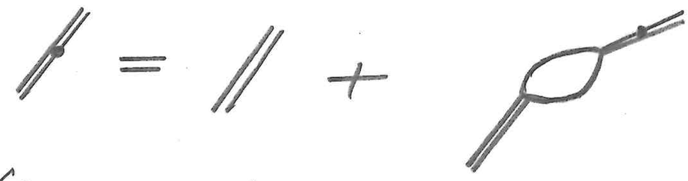
$$\longleftrightarrow \quad i G(x-y)$$

In fact we can show

$$(\partial_x^2 + m^2) \langle 0 | T\{ \varphi_x \varphi_y \} | 0 \rangle = -i \delta^4_{x-y}$$

$$(\partial_x^2 + m^2) \varphi_x = 0$$ as eq. of motion.

$$\langle 0 | T\{ \varphi_x \varphi_y \} | 0 \rangle \longleftrightarrow \frac{-i}{\partial_x^2 + m^2} \delta^4_{x-y}$$

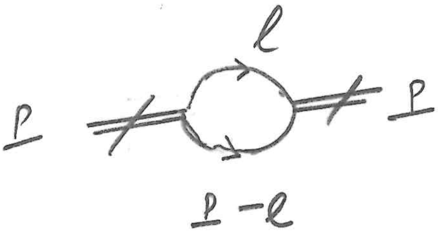


$$iG = iG_0 + iG_0(-i\Sigma)\, iG$$

$$\Rightarrow \quad G^{-1} = G_0^{-1} - \Sigma$$

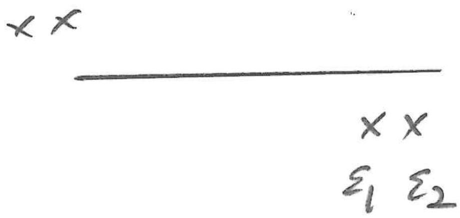
$$-i\Sigma(p) = \int \frac{d^4 l}{(2\pi)^4}\, iD_{l_1}\, iD_{l_2}\, (-ig)^2$$

$$l_1 = l$$
$$l_2 = p - l$$



$$\frac{i}{l_1^2 - m^2 + i\delta} \qquad \frac{i}{l_2^2 - m^2 + i\delta}$$

l_1^0 ; l_2^0



Φ_2

if $p^0 > 0$

$$\mathrm{Im}\,\Sigma_{p^0, \vec{p}} = -\frac{1}{2} g^2 \int \frac{d^3 l_1}{(2\pi)^3} \frac{d^3 l_2}{(2\pi)^3} \frac{1}{2E_1\, 2E_2} (2\pi)\, \delta_{p^0 - E_1 - E_2}\, (2\pi)^3 \delta^3_{\vec{p} - \vec{l}_1 - \vec{l}_2}$$

$$= -\frac{1}{2} g^2\, \Phi_2(s)$$

$$G^{-1} = G_0^{-1} - \Sigma$$

$$\Rightarrow \quad G(p^0, \vec{p}) = \frac{1}{p^{0\,2} - \vec{p}^{\,2} - m^2 - \Sigma_R - i\Sigma_I}$$

CM:
$\vec{p} = 0$
$p^0 \to E$

$$= \frac{1}{E^2 - \bar{m}^2 - \underline{i\Sigma_I(E)}}$$

eff. mass of the resonance

$$\bar{m}^2 = m^2 + \Sigma_R(E)$$

$$+ iE\gamma(E)$$

$$\longrightarrow \frac{1}{E^2 - \bar{m}^2 + iE\gamma(E)}$$

around $E \simeq \bar{m}$

$$\simeq \frac{1}{2\bar{m}} \left(\frac{1}{E - \bar{m} + i\frac{1}{2}\gamma(E)} \right)$$

$$\frac{1}{E - m - \Delta}$$

$\Delta_R \leftrightarrow E$ shift

$-2\Delta_I \leftrightarrow$ width.

We have just seen a specific case of a general Cutkosky rule based on unitarity

$-2\,\mathrm{Im}$  $= \int d\Phi \; \Big|$  $\Big|^2$

2-body final states

g^2

$$-2\,\mathrm{Im}\,\Sigma = 2\sqrt{s}\,\Gamma(s) = g^2\,\Phi_2 .$$

$$\Gamma(s) = \frac{1}{2\sqrt{s}} \int d\Phi_2 \; g^2$$

Relativistic analog of Golden Rule

$$\sum_f |V_{fi}|^2 \; 2\pi\,\delta(E_f - E_i)$$