Green's function in Maxwell's eq.

$$\nabla \cdot \vec{E} = \rho_{\vec{x}}$$

$\vec{E} = -\nabla \phi$ electrostatic potential ϕ

$$-\nabla^2 \phi_{\vec{x}} = \rho_{\vec{x}}$$

$\Rightarrow$ how to realize / advanced notation

$$\phi_{\vec{x}} = \frac{-1}{\nabla^2_{\vec{x}}} \rho_{\vec{x}} \quad ?$$

$$= \int d^3x' \underbrace{\frac{-1}{\nabla^2_{\vec{x}}} \delta_{\vec{x}\vec{x}'}}_{\text{Green's function } G_{\vec{x}\vec{x}'}} \rho_{\vec{x}'}$$

$G_{\vec{x}\vec{x}'}$ satisfies $\quad -\nabla^2_{\vec{x}} G_{\vec{x}\vec{x}'} = \delta_{\vec{x}-\vec{x}'}$

s.t. $\quad -\nabla^2_{\vec{x}} \phi_{\vec{x}} = \int d^3x' \delta_{\vec{x}\vec{x}'} \rho_{\vec{x}'} = \rho_{\vec{x}}$ //

$$G_{\vec{x}\vec{x}'} = \frac{-1}{\nabla^2_{\vec{x}}} \delta_{\vec{x}\vec{x}'} = \int \frac{d^3k'}{(2\pi)^3} \frac{1}{\vec{k}'^2} e^{i\vec{k}' \cdot (\vec{x}-\vec{x}')}$$

$$= \text{FT of } \frac{1}{\vec{k}^2}$$

$$\frac{1}{\vec{q}^2+m^2} \longleftrightarrow \frac{1}{4\pi r} e^{-mr}$$

$$\frac{1}{\vec{q}^2} \longleftrightarrow \frac{1}{4\pi\,|\vec{x}-\vec{x}'|}$$

$$G_{\vec{x}\vec{x}'} = \frac{1}{4\pi\,|\vec{x}-\vec{x}'|} = \frac{-1}{\nabla_x^2}\,\delta_{\vec{x}-\vec{x}'}$$

obviously

$$-\nabla^2 G_{\vec{x}\vec{x}'} = \delta_{\vec{x}-\vec{x}'}$$

understand this

$$\nabla_{\vec{x}}^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta_{\vec{x}-\vec{x}'}$$

$$\varphi_{\vec{x}} = \int d^3x' \frac{1}{4\pi\,|\vec{x}-\vec{x}'|} \underline{\rho_{\vec{x}'}}$$

if $\rho_{\vec{x}'} = \delta_{\vec{x}'-\vec{a}}\, Q$

$$\varphi_{\vec{x}} \longrightarrow \frac{Q}{4\pi\,|\vec{x}-\vec{a}|}$$

$$\vec{E} = -\nabla\varphi_{\vec{x}} = \frac{Q}{4\pi R^2}\hat{R}$$

$$Q = \int d^3x\, Q\, \delta^3_{\vec{x}-\vec{a}} = \int d\vec{A}\cdot\vec{E} = \int d^3x\, \frac{Q}{4\pi}\left[-\nabla^2\frac{1}{|\vec{x}-\vec{a}|}\right]$$

$$\nabla^2 \frac{1}{|\vec{x}-\vec{a}|} = -4\pi\,\delta_{\vec{x}-\vec{a}} \quad //$$

②

$$\frac{1}{\vec{p}^{2}+m^{2}} \longrightarrow \frac{1}{-\nabla^{2}+m^{2}}$$

$$\left[-\nabla^{2}+m^{2}\right] G_{\vec{x}\vec{x}'} = \delta_{\vec{x}\vec{x}'}$$

$$G_{\vec{x}\vec{x}'} = \frac{1}{-\nabla^{2}+m^{2}} \delta_{\vec{x}\vec{x}'}$$

$$= \int \frac{d^{3}k}{(2\pi)^{3}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \frac{1}{\vec{p}^{2}+m^{2}}$$

$$= \frac{e^{-m|\vec{x}-\vec{x}'|}}{4\pi |\vec{x}-\vec{x}'|}$$

//

$$\left[\partial_{t}^{2}-\nabla^{2}+m^{2}\right] G_{xx'} = -\delta^{4}_{x-x'}$$

$$\Rightarrow G_{xx'} = \int \frac{d^{4}p}{(2\pi)^{4}} e^{-ip\cdot(x-x')} \frac{1}{p^{2}-m^{2}+i\delta}$$

$$= \frac{-1}{\partial_{x}^{2}+m^{2}} \delta^{4}_{xx'}$$

//

$$\frac{1}{p^{0\,2} - \vec{p}^{\,2} - m^2} = \frac{1}{p^{0\,2} - \varepsilon_{\vec{p}}^{\,2}}$$

is QFT propagator

$\longrightarrow$ my favorite Q

QFT = QM + special relativity

QM particle can be off-shell $\Leftarrow$ $p^0 = \varepsilon_{\vec{p}}$ on shell

Δt borrow energy $p^0 \neq \varepsilon_{\vec{p}}$ for a moment

then what is

$$\frac{1}{p^{0\,2} - \varepsilon_{\vec{p}}^{\,2}} \quad ?$$

eqn. of motion $\Rightarrow$ not like classical.

QFT eqn. of motion $\Leftarrow$ can borrow energy

Path integral $\Rightarrow$ plausible way to understand it