Basic QFTs

Free Scalar Fields

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2$$

Equation of Motion:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} = \frac{\partial \mathcal{L}}{\partial \varphi}$$

$$\partial^2 \varphi = -m^2\varphi$$

$$(\partial^2 + m^2)\varphi = 0$$

$$\pi_x := \frac{\partial \mathcal{L}}{\partial \partial_t \varphi_x} = \partial_t \varphi_x$$

$$[\varphi_x, \pi_{x'}] = i\,\delta^3_{\vec{x}\vec{x}'} \quad \text{at} \quad t = t'$$

Realization #1

$$\varphi_x = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \left\{ e^{-ik\cdot x} a_{\vec{k}} + e^{ik\cdot x} a^+_{\vec{k}} \right\}$$

$$k^0 = E_k$$

$$\pi_x = \int \frac{d^3k}{(2\pi)^3} \frac{-i}{2} \left\{ e^{-ik\cdot x} a_{\vec{k}} - e^{ik\cdot x} a^+_{\vec{k}} \right\}$$

convention:

$$\langle 0 | a_{\vec{k}} a^{\dagger}_{\vec{k}'} | 0 \rangle = (2\pi)^3 \delta^3_{\vec{k}-\vec{k}'} \, 2E_{\vec{k}}$$

$$\langle 0 | T\{ \varphi_x \varphi_y \} | 0 \rangle$$

$$= \theta(x^0 - y^0) \langle 0 | \varphi_x \varphi_y | 0 \rangle +$$

$$\theta(y^0 - x^0) \langle 0 | \varphi_y \varphi_x | 0 \rangle$$

$$= \theta(x^0 - y^0) \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{1}{2E_1} \frac{1}{2E_2} e^{-ik_1 \cdot x} e^{+ik_2 \cdot y}$$

$$(2\pi)^3 \delta^3_{\vec{k}_1 - \vec{k}_2} \, 2E_1$$

$$+$$

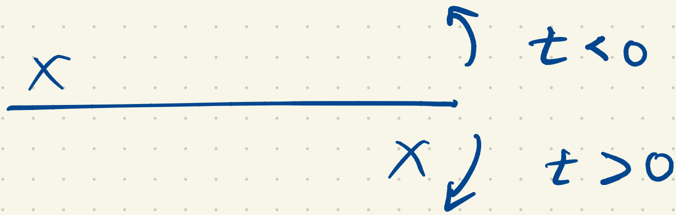
$$\theta(y^0 - x^0) \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{1}{2E_1} \frac{1}{2E_2} e^{-ik_2 \cdot y} e^{ik_1 \cdot x}$$

$$(2\pi)^3 \delta^3_{\vec{k}_1 - \vec{k}_2} \, 2E_1$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} e^{-iE_k |x^0 - y^0|}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\delta} e^{-ik\cdot(x-y)} \longleftrightarrow iG_2$$

LK^0



extra factor -1

$$iG_2 = \langle 0|T\{\varphi_x \varphi_y\}|0\rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\epsilon} e^{-ik\cdot(x-y)}$$

//

Realization #2

$$(\partial_x^2+m^2)\, G_2(x,y) = -\delta^4_{x-y} \quad \text{or}$$

$$(\partial_x^2+m^2)\langle 0|T\{\varphi_x\varphi_y\}|0\rangle = -i\delta^4_{x-y}$$

How to see that?

$$(\partial_x^2+m^2)\,\{\theta_{x^0-y^0}\langle\varphi_x\varphi_y\rangle+\theta_{y^0-x^0}\langle\varphi_y\varphi_x\rangle\}$$

$$\partial_{t_x}^2\left(\theta_{x^0-y^0}\,\varphi_x\varphi_y\right)$$

$$=\partial_{t_x}\left(\delta_{x^0-y^0}\,\varphi_x\varphi_y+\theta_{x^0-y^0}\,\dot{\varphi}_x\varphi_y\right)$$

$$=\delta_{x^0-y^0}\,\dot{\varphi}_x\varphi_y+\underline{\theta_{x^0-y^0}\,\ddot{\varphi}_x\varphi_y}$$

$\longrightarrow 0$ when combined with $(-\partial_x^2+m^2)$ on φ_x

thus:

$$LHS\longrightarrow\delta_{x^0-y^0}\langle\dot{\varphi}_x\varphi_y-\varphi_y\dot{\varphi}_x\rangle$$

$$=\underline{\delta_{x^0-y^0}}\langle[\dot{\varphi}_x,\varphi_y]\rangle$$

enforcing equal time!

$$\Rightarrow(\partial_x^2+m^2)\langle 0|T\{\varphi_x\varphi_y\}|0\rangle=-i\,\delta^4_{x-y}\;//$$

Realization #3

Functional Method

$$Z[j] = \int D\varphi \, e^{i\int\left(-\frac{1}{2}\varphi(\partial^2+m^2)\varphi + j\varphi\right)}$$

$$= \frac{1}{\sqrt{\det(\partial^2+m^2)}} \, e^{\int j \frac{-1}{4i\,\frac{1}{2}(\partial^2+m^2)} j}$$

$$= \frac{1}{\sqrt{\det(\partial^2+m^2)}} \, e^{i\frac{1}{2}\int j \frac{1}{\partial^2+m^2} j}$$

$$W[j] = -i \ln Z[j]$$

$$\longrightarrow \frac{1}{2}\int j \frac{1}{\partial^2+m^2} j$$

$$\frac{-\delta^2 W}{\delta j_x \, \delta j_y} = -\frac{1}{\partial_x^2+m^2}\delta^4_{xy}$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{1}{k^2-m^2+i\delta}$$

i.e. $\quad G_2 \longleftrightarrow -\frac{\delta^2 W}{\delta j_x \, \delta j_y} = -i\langle T\{\varphi_x \varphi_y\}\rangle$

or the Γ-picture:

Recall

$$W[j] = \frac{1}{2}\int j \frac{1}{\partial^2+m^2} j$$

$$\frac{\delta W}{\delta j_x} = \frac{1}{\partial_x^2+m^2} j_x \quad \leftrightarrow \quad \bar{\varphi}_x$$

$$j_x \to (\partial_x^2+m^2)\,\bar{\varphi}_x$$

$$\Gamma := W - \int j\,\bar{\varphi}$$

$$= \frac{1}{2}\int \bar{\varphi}(\partial^2+m^2)\bar{\varphi} - \int \bar{\varphi}(\partial^2+m^2)\bar{\varphi}$$

$$= -\frac{1}{2}\int \bar{\varphi}(\partial^2+m^2)\bar{\varphi}$$

$$\frac{\delta^2\Gamma}{\delta\bar{\varphi}_x\delta\bar{\varphi}_y} = -(\partial_x^2+m^2)\,\delta^4_{xy}$$

$$\longrightarrow \; k^2 - m^2 \quad \text{in } k\text{-space}$$

$$\leftrightarrow \; G_2^{-1}$$

With functional method

$\longrightarrow$ saddle

$-W_2 = G_2$ is an interacting QFT !

Γ_2^{-1} $\Longrightarrow$ Schwinger - Dyson

Equation !