SU(2) chiral symmetry

$SU_V(2)$

$$q \to e^{-i\alpha_V^a \tau^a} q$$

$\hookrightarrow$ 4-spinor

$\binom{u}{d}$ doublet

$$\mathcal{L}_0 = \bar{q}(i\partial\!\!\!/ - \hat{m})q$$

$$\hat{m} = \begin{bmatrix} m_u & \\ & m_d \end{bmatrix}$$

$$H_0 = -i\bar{q}\vec{\gamma}\cdot\nabla q + \bar{q}\hat{m}q$$

$$\Delta\mathcal{L}_0 \to \bar{q}e^{i\alpha_V^a\tau^a}(i\partial\!\!\!/ - \hat{m})e^{-i\alpha_V^b\tau^b}q$$

$$- \bar{q}(i\partial\!\!\!/ - \hat{m})q$$

$$\simeq -i\alpha_V^a \, \bar{q}[\tau^a, \hat{m}]q$$

$$\Delta\mathcal{L}_0^a \to \Big\{ i\,\tfrac{1}{2}(m_u - m_d)(\bar{u}d - \bar{d}u),$$

$$+\tfrac{1}{2}(m_u - m_d)(\bar{u}d - \bar{d}u),$$

$$0 \Big\}$$

note

$$\Delta H_0^a = -\Delta\mathcal{L}_0^a$$

$SU_V(2)$ isospin.

$\Delta \mathcal{L} \neq 0$ if $m_u \neq m_d$.

in fact Noether current does <u>not</u> conserve

$$\Longrightarrow \quad j_V^{a\mu} = \bar{q}\gamma^\mu \tau^a q$$

&

$$\partial_\mu j_V^{a\mu} \to \partial_t j_V^{a0} = \Delta \mathcal{L}^a$$

$$Q_V^a = \int d^3x \, j_V^{a0}$$

$$= \int d^3x \left(\bar{q}\gamma^0 \tau^a q\right) \quad //$$

it follows that

$$[Q_V^a, H] = i\partial_t Q^a = i\Delta \mathcal{L}^a = -i\Delta H^a$$

$$= \int d^3x \, \bar{q}\,[\tau^a, \hat{m}]\, q \quad //$$

For $SU_A(2)$

$$q \to e^{-i\gamma_5 \alpha^a_{AV} \tau^a} q$$

note $\gamma_5^\dagger = \gamma_5$

$$\Delta \mathcal{L}_0 \simeq i \alpha^a_{AV} \quad \bar{q} \gamma_5 \{\tau^a, \hat{m}\} q$$

$$j^{a\mu}_{AV} = \bar{q} \gamma^\mu \gamma_5 \tau^a q$$

$$\partial_\mu j^{a\mu}_{AV} = \Delta \mathcal{L}^a$$

$$Q^a_{AV} = \int d^3x \quad \bar{q} \gamma^\mu \gamma_5 \tau^a q$$

$$[Q^a_{AV}, H] = i\partial_t Q^a_{AV} = -\int d^3x \; \bar{q} \gamma_5 \{\tau^a, \hat{m}\} q$$

$\longrightarrow 0$ only if

$\hat{m} \to 0$

massless case.

justified for u, d quarks

can we verify this directly?

$$Q^a_{AV} = \int d^3x \; \bar{q}_x \gamma^0 \gamma_5 \tau^a q_x$$

$$H = \int d^3x \; \mathcal{H}$$

ng $\{q^a_{\vec{x}}, q^{b\dagger}_{\vec{y}}\} = \delta^3_{\vec{x}-\vec{y}} \, \delta_{ab}$ equal ~~time~~

0 otherwise

$$[Q^a_{AV}, H]$$

other commutes w Q^a_{AV}

$$= \int\int \left[\bar{q}_x \gamma^0 \gamma_5 \tau^a q_{x'}, \; \bar{q}_y \hat{m} q_y + \cdots\right]$$

$$= \int\int \bar{q} \gamma^0 \gamma_5 \tau^a q \; \bar{q} \hat{m} q$$

$$\to \int \bar{q} \left(\gamma^0 \gamma_5 \gamma^0 \tau^a \hat{m} - \hat{m} \gamma_5 \tau^a\right) q$$

$$= -\int \bar{q} \gamma_5 \{\tau^a, \hat{m}\} q$$

$$= -i \, \Delta H^a$$

//

In fact, we can see the PML trick

$$\underline{i\partial_t} Q^a = [Q^a, \underline{H}] \qquad \text{time evolution}$$

$$\underline{i\partial_{\alpha_a}} H = [H, \underline{Q^a}] \qquad \text{symmetry transform}$$

$$[Q^a, H] \Longleftrightarrow -i\,\underline{\Delta H}^a = i\,\underline{\Delta \mathcal{L}}^a$$

running a chiral transform

SSB

$$\mathcal{U} = e^{-i\alpha^a Q^a}$$

if $\hat{m} \to 0$ s.t. $[Q^a, H] = 0$

$$|0'\rangle = \mathcal{U}|0\rangle$$

$$\langle 0'|H|0'\rangle = \langle 0|\mathcal{U}^\dagger H \mathcal{U}|0\rangle$$

$$= \langle 0|H|0\rangle$$

$$\underline{E'} = \underline{E}$$

Normally $\quad |0'\rangle = |0\rangle \quad$ or $\quad Q|0'\rangle = 0$

SSB $\quad Q|0'\rangle \neq 0$

this state is so special

$$H\,Q|0'\rangle = E_0\,Q|0'\rangle$$

does not cost energy to excite $\longrightarrow$ Goldstone's Bosons.

$\hat{O} \rightarrow [\hat{O}, Q] \neq 0$

$\langle 0' | \hat{O} | 0' \rangle \neq \langle 0 | \hat{O} | 0 \rangle$

$\longrightarrow$ can be used to probe vac. structure

e.g.

$$\bar{q}q \longrightarrow \bar{q}q - \alpha^a_{AV} \left(2i \, \bar{q} \gamma_5 \tau^a q \right)$$

$$\left(-2i \, \bar{q} \gamma_5 \tau^a q \right) = [\bar{q}q, Q^a_{AV}]$$

$$2 \, \bar{q} \gamma_5 \tau^a q = [\bar{q}q, Q^a_{AV}]$$

$$Q^a_{AV} = \int d\vec{x} \, \bar{q} \gamma^0 \gamma_5 \tau^a q$$

$\langle 0' | \bar{q} q | 0' \rangle$ is one choice of order parameter

SSB

$[Q, H] = 0$ sym. is respected

but but

$Q|0\rangle \neq 0$ NOT by vac.

many consequences

s.t. as massless Goldstone bosons.

G.O.R.

→ but current is conserved

$m_q \neq 0$ → partially

PCAC

the emergence of $\sigma, \vec{\pi}$

$$\bar{q}q \xrightarrow{AV} \bar{q}q - \alpha^a \, 2i\, \bar{q}\gamma_5 \tau^a q$$

$$2i\,\bar{q}\gamma_5\tau^a q \xrightarrow{AV} 2i\,\bar{q}\gamma_5\tau^a q + \alpha^a \bar{q}q$$

$\Rightarrow$ like a rotation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x' \simeq x - \theta y$$
$$y' \simeq y + \theta x$$

$(\bar{q}q)^2 + (2i\,\bar{q}\gamma_5\tau^a q)^2$ is invariant

If we introduce

$$\left.\begin{aligned} \sigma &\xrightarrow{AV} \sigma + \alpha^a_{AV}\pi^a \\ \pi^a &\xrightarrow{AV} \pi^a - \alpha^a_{AV}\sigma \end{aligned}\right\} \text{ opposite rotation}$$

s.t.

$\bar{q}(\sigma - 2i\gamma_5 \vec{\tau}\cdot\vec{\pi})q$ is invariant

i.e. $\bar{q}'(\sigma' - 2i\gamma_5\vec{\tau}\cdot\vec{\pi}')q' = \bar{q}(\sigma - 2i\gamma_5\vec{\tau}\cdot\vec{\pi})q$

Similarly $(\sigma^2+\vec{\pi}^2)$, $(\bar{q}q)^2+(2i\,\bar{q}\gamma_5\vec{\tau}q)^2$ are invariant!

Summarize

how composite operators rotate?

$$\bar{q}q \xrightarrow{V} \bar{q}q$$

$\vec{\alpha}_V \times (2i\,\bar{q}\gamma_5\vec{\tau}q)$

$$2i\,\bar{q}\gamma_5\tau^a q \xrightarrow{V} 2i\,\bar{q}\gamma_5\tau^a q + \varepsilon^{abc}\alpha_V^b\, 2i\,\bar{q}\gamma_5\tau^c q$$

$$\bar{q}q \xrightarrow{AV} \bar{q}q - \alpha_{AV}^a\,(2i\,\bar{q}\gamma_5\tau^a q)$$

$$2i\,\bar{q}\gamma_5\tau^a q \xrightarrow{AV} 2i\,\bar{q}\gamma_5\tau^a q + \alpha_{AV}^a\,\bar{q}q$$

$$\sigma \xrightarrow{V} \sigma$$

$$\vec{\pi} \xrightarrow{V} \vec{\pi} + \vec{\alpha}_V \times \vec{\pi}$$

$$\sigma \xrightarrow{AV} \sigma + \vec{\alpha}_{AV}\cdot\vec{\pi}$$

$$\vec{\pi} \xrightarrow{AV} \vec{\pi} - \vec{\alpha}_{AV}\,\sigma$$

Note that

$$\bar q\,(\sigma - \vec\pi\cdot 2i\gamma_5\vec\tau)\,q$$

$$\xrightarrow{V}\ \bar q\,(\sigma - \vec\pi\cdot 2i\gamma_5\vec\tau)\,q$$

where $\vec\pi_1 = 2i\,\bar q\gamma_5\vec\tau q$

$$-\ \Delta_a\vec\pi\cdot(2i\,\bar q\gamma_5\vec\tau q) \longrightarrow -(\vec\alpha_V\times\vec\pi)\cdot\vec\pi_1$$

$$-\ \vec\pi\cdot\Delta(2i\,\bar q\gamma_5\vec\tau q) \longrightarrow -\vec\pi\cdot(\vec\alpha_V\times\vec\pi_1)$$

$\Downarrow$

they cancel!

it is invariant under V

$+\ AV$

Current Algebra

$$Q_V^a = \int \bar q\gamma^0\tau^a q$$

$$Q_{AV}^a = \int \bar q\gamma^0\gamma_5\tau^a q$$

we can work out

$$[Q_V^a, Q_V^b] = i\varepsilon^{abc}Q_V^c$$

$$[Q_{AV}^a, Q_{AV}^b] = i\varepsilon^{abc}Q_V^c$$

$$[Q_V^a, Q_{AV}^b] = i\varepsilon^{abc}Q_{AV}^c$$

if we define

$$Q_{\substack{L\\R}} := \frac{1}{2}\left(Q_V \pm Q_{AV}\right)$$

$$[Q_L^a, Q_L^b] = i\varepsilon^{abc} Q_L^c$$

$$[Q_R^a, Q_R^b] = i\varepsilon^{abc} Q_R^c$$

$$[Q_L^a, Q_R^b] = 0 \quad //$$

$\Longrightarrow$ L, R decoupled !

$\longrightarrow$ good language to understand the symmetry

$$q_{\substack{L\\R}} = \underbrace{\frac{1}{2}\left(\mathbb{I} \pm \gamma_5\right)}_{\hat{P}_{\substack{L\\R}}} q$$

note:

$$\bar{q}_L \to (\hat{P}_L q)^\dagger \gamma^0 = \bar{q}\hat{P}_R$$

$$\hat{P}_L \hat{P}_R = \hat{P}_R \hat{P}_L = 0$$

$$\bar{q}q = \bar{q}_L q_R + \bar{q}_R q_L \longrightarrow \text{mix}$$

$$\bar{q}\gamma^\mu q = \bar{q}_L\gamma^\mu q_L + \bar{q}_R\gamma^\mu q_R \longrightarrow \times \text{ mix}$$

$L, R \leftrightarrow V, AV$ via

$$U_L \leftrightarrow e^{-i(\vec{\alpha}_V + \vec{\alpha}_{AV})\cdot\vec{\tau}}$$

$$U_R \leftrightarrow e^{-i(\vec{\alpha}_V - \vec{\alpha}_{AV})\cdot\vec{\tau}}$$

for V

$$U_L \to e^{-i\vec{\alpha}_V\cdot\vec{\tau}}$$

$$U_R = U_L = e^{-i\vec{\alpha}_V\cdot\vec{\tau}}$$

for AV

$$U_L \to e^{-i\vec{\alpha}_{AV}\cdot\vec{\tau}}$$

$$U_R \to e^{+i\vec{\alpha}_{AV}\cdot\vec{\tau}}$$

$$= U_L^{+}$$

$$\Sigma = \sigma\, \mathbb{I}_2 + \vec{\pi}\cdot 2i\vec{\tau}$$

$$P_{L/R} = \frac{1}{2}(I \pm \gamma_5)$$

//

$$\Sigma \longrightarrow U_L \Sigma U_R^{+}.$$

// s.t.

$$\overline{q_L}\,\Sigma\, q_R + \overline{q_R}\,\Sigma^{+} q_L$$ is invariant by construction

$$\to \overline{q_L}(\sigma + i\, 2\vec{\pi}\cdot\vec{\tau})\, q_R + \overline{q_R}\,(\sigma - i\,\vec{\pi}\cdot 2\vec{\tau})\, q_L$$

$$= \sigma\,\overline{q}q + \overline{q}\left(-\vec{\pi}\cdot 2i\vec{\tau}\right)\frac{(P_L - P_R)}{\gamma_5}\, q$$

$$\to \overline{q}\left(\sigma - i\vec{\pi}\cdot 2\vec{\tau}\,\gamma_5\right) q$$ //

Now how $\sigma, \vec{\pi}$ transform ?

$$\Sigma \xrightarrow{V} U_V \Sigma U_V^+$$

$$= U_V (\sigma \mathbb{I}_2 + \vec{\pi} \cdot 2i\vec{\tau}) U_V^+$$

$$\approx \sigma \mathbb{I}_2 + i(\vec{\pi} + \vec{\alpha}_V \times \vec{\pi}) \cdot 2\vec{\tau}$$

//

thus

$$\sigma \rightarrow \sigma$$

$$\vec{\pi} \rightarrow \vec{\pi} + \vec{\alpha}_V \times \vec{\pi}$$

//

$$\Sigma \xrightarrow{AV} U_{AV} \Sigma U_{AV}$$

$$\longrightarrow \sigma e^{-2i\vec{\alpha}_{AV} \cdot \vec{\tau}} + U_{AV}\, i\vec{\pi} \cdot 2\vec{\tau}\, U_{AV}$$

$$\simeq \sigma - 2i\vec{\alpha}_{AV} \cdot \vec{\tau}\sigma + i\vec{\pi} \cdot 2\vec{\tau}$$

$$+ 2\alpha^b_{AV} \pi^a \{\tau^b, \tau^a\}$$

$$= (\sigma + \vec{\alpha}_{AV} \cdot \vec{\pi}) +$$

$$(\vec{\pi} - \sigma\vec{\alpha}_{AV}) \cdot 2i\vec{\tau}$$

//

$$\sigma \rightarrow \sigma + \vec{\alpha}_{AV} \cdot \vec{\pi}$$

$$\vec{\pi} \longrightarrow \vec{\pi} - \vec{\alpha}_{AV} \sigma.$$

//

last, but not least.

$$\sigma = \operatorname{tr}\Sigma / \operatorname{tr} \mathbb{I}_2$$

$$\Sigma \to U_L \Sigma U_R^\dagger$$

for V

$$\Sigma \to U_V \Sigma U_V^\dagger$$

$$\operatorname{tr}\Sigma \to \operatorname{tr}\Sigma$$

σ is invariant

~~for~~ AV

$$\Sigma \to U_{AV} \Sigma U_{AV}$$

$\operatorname{tr}\Sigma$ is <u>NOT</u> invariant one of the key result from current algebra.

no sum.

$$m_\pi^2 = \frac{-1}{f_\pi^2} \langle 0 | [Q^a_{AV}, [Q^a_{AV}, H]] | 0 \rangle$$

one can derive $\rightarrow$ $[Q^a_{AV}, [Q^a_{AV}, H]] = \int \bar{q} \hat{m} q$

— G.O.R.

$$m_\pi^2 = \frac{-1}{f_\pi^2} \left(m_u \langle \bar{u} u \rangle + m_d \langle \bar{d} d \rangle \right)$$

$-(250\ \mathrm{MeV})^3$

93 MeV

few MeV