Hadrons: Homework 2

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Q1 Collider Design

Questions:

Compute and estimate the following:

a) Relativistic Acceleration

Compute the kinetic energy required to accelerate a particle:

- a proton from $v = 0$ to $v = 0.99c$ and $v = 0.99c$ to $v = 0.999c$.
- an electron for the same speeds.

b) Synchrotron Radiation Power

Estimate the power needed to maintain a stored beam in the storage ring:

- a 6.8 TeV proton beam in the LHC ($R_{\mathrm{LHC}} = 4.3\mathrm{km}$, 1240 bunches, 1.6×10^{11} protons per bunch).
- an 18 GeV electron beam in the EIC ($R_{\mathrm{EIC}} = 614\mathrm{m}$, $I = 0.26\mathrm{A}$).

Use the radiation formula:

$$P_{\mathrm{rad}} = \frac{e^2 a^2}{6\pi}\gamma^4, \quad a = \frac{v^2}{R}, \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

c) Collider vs Fixed Target

Find the proton beam energy required to reach a center-of-mass energy $\sqrt{s} = 14$ TeV for:

- colliding beams.
- a fixed-target experiment.

Explain why the fixed-target setup is inefficient compared to the collider.

Solutions

a) To compute the kinetic energy, we can start with the Work-Energy theorem to compute the necessary force to accelerate the particle.

$$W = \int \vec{F} \cdot \mathrm{d}\vec{x}\,.$$

The catch here is that we are going to use the relativistic 3-force, with $\vec{F} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m \vec{v}\right)$; with γ being the Lorentz factor $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, a function of speed v. The result of this integral is precisely the amount of Kinetic energy necessary to accelerate the particle. We also assume that the force is parallel to the final direction of motion of the particle.

$$
\begin{aligned}
K &= \int \vec{F} \cdot \mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m\vec{v})\,\mathrm{d}x\,, \\
&= \int \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m v\right)\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t\,, \\
&= \int_{v_1}^{v_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m v\right) v\,\mathrm{d}t\,.
\end{aligned}
$$

Now integrate by parts using $\int u\,\mathrm{d}w = uw - \int w\,\mathrm{d}u$. We can identify $u = v$, $\mathrm{d}u = \frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = \mathrm{d}v$, and $\mathrm{d}w = \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m v\right)\mathrm{d}t$, $w = \gamma m v$.

$$
\begin{aligned}
K &= \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}}\Bigg|_{v_1}^{v_2} - \int_{v_1}^{v_2} \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\,\mathrm{d}v\,, \\
\text{Let } z &= 1 - \frac{v^2}{c^2} \qquad \mathrm{d}z = -\frac{2v}{c^2}\,\mathrm{d}v\,, \\
&= \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}}\Bigg|_{v_1}^{v_2} + \frac{mc^2}{2}\int \frac{\mathrm{d}z}{\sqrt{z}}, \\
&= \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}}\Bigg|_{v_1}^{v_2} + mc^2\sqrt{z}, \\
&= \left[\frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} + mc^2\sqrt{1-\frac{v^2}{c^2}}\right]\Bigg|_{v_1}^{v_2}, \\
&= \left[\frac{mv^2 + mc^2\left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}\right]\Bigg|_{v_1}^{v_2}, \\
&= \left[\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\right]\Bigg|_{v_1}^{v_2}, \\
&= mc^2\left[\gamma(v)\right]\Big|_{v_1}^{v_2}, \\
&= mc^2\left[\gamma(v_2) - \gamma(v_1)\right].
\end{aligned}
$$

Now, with the mass of the proton being $m_p = 0.938$ GeV$/c^2$, we can compute the Kinetic energy for each case.

- From $v = 0$ to $v = 0.99c$:

$$
K = (0.938\text{ GeV})\left[\frac{1}{\sqrt{1-(0.99)^2}} - \frac{1}{\sqrt{1-(0)^2}}\right] = 5.711305\text{ GeV}.
$$

- From $v = 0.99c$ to $v = 0.999c$:

$$K = (0.938 \text{ GeV}) \left[\frac{1}{\sqrt{1-(0.999)^2}} - \frac{1}{\sqrt{1-(0.99)^2}} \right] = 14.330257 \text{ GeV}.$$

It takes nearly 2.5 times more energy to accelerate a proton from $v = 0.99c$ to $v = 0.999c$ than it takes to accelerate it to $v = 0.99c$ from rest. For an electron ($m_e = 0.5110 \text{Mev}/c^2$) this ratio will be true, but with different energies:

- From $v = 0$ to $v = 0.99c$:

$$K = (0.5110 \text{ MeV}) \left[\frac{1}{\sqrt{1-(0.99)^2}} - \frac{1}{\sqrt{1-(0)^2}} \right].$$

$$\boxed{K = 3.1113 \text{ MeV}}.$$

- From $v = 0.99c$ to $v = 0.999c$:

$$K = (0.5110 \text{ MeV}) \left[\frac{1}{\sqrt{1-(0.999)^2}} - \frac{1}{\sqrt{1-(0.99)^2}} \right].$$

$$\boxed{K = 7.8067 \text{ MeV}}.$$

b) We start with the radiation formula, and we manipulate it as following:

$$\begin{aligned} P_{\text{rad}} &= \frac{e^2 a^2 \gamma^4}{6\pi} = \frac{e^2 \gamma^4 v^4}{6\pi R^2}, \\ &= \frac{e^2 \left(m^2 \gamma^2 v^2\right)^2}{6\pi R^2 m^4} = \frac{e^2 \left(E^2 - m^2\right)^2}{6\pi R^2 m^4}. \end{aligned}$$

Where we first substituted $a = \frac{v^2}{R}$, and then we combined the quantity $\gamma m v$ into the relativistic 3-momentum p. Then, we used the energy-momentum relation $E^2 = m^2 + p^2$. And now, we can consider the ultra-relativistic regime, where $E \gg m$. And drop the mass term. Now, consider the fine-structure constant $\alpha = \frac{e^2}{4\pi}$, and substitute it in the expression:

$$P_{\text{rad}} = \frac{e^2 E^4}{6\pi R^2 m^4} = \frac{2\alpha E^4}{3 m^4 R^2}.$$

Now, the energy lost per turn can be computed using this result:

$$\Delta U_{\text{per turn}} = P_{\text{rad}} \Delta t = P_{\text{rad}} \frac{2\pi R}{c}.$$

These result consider the radiated power and the energy lost per turn for a *single particle*. We can then solve for the Power to maintain a stored beam in the EIC using electrons. To compute for the whole beam, we need to basically count the number of electrons. Because we are given a current I we can just multiply ΔU by it and divide by the charge of a single electron:

$$P = \Delta U \frac{I}{e} = \frac{2\alpha E^4}{3m^4R^2}\frac{2\pi RI}{ce}.$$

Now we can start substituting known values. We will use :

$$\begin{aligned} m_e &= 0.000511 \text{ GeV}, \quad E = 18 \text{ GeV}, \\ I &= 0.26 \text{ A}, \quad |e| = 1.6 \times 10^{-19} \text{ C}, \\ R &= 614 \text{ m} = 3111.138 \times 10^{15}\text{GeV}^{-1}, \\ 1\text{GeV}^2 &= 2.43 \times 10^{14} \text{ W}, \alpha = \frac{1}{137}. \end{aligned}$$

Leading to the final result:

$$\begin{aligned} P = &\left(\frac{2\,(18 \text{ GeV})^4}{(137)3\,(0.000511 \text{ GeV})^4 \left(3111.138 \times 10^{15}\text{GeV}^{-1}\right)^2} \frac{2.43 \times 10^{14} \text{ W}}{(1 \text{ GeV})^2} \right) \times \\ &\times \left(\frac{2\pi(614 \text{ m})(0.26 \text{ A})}{(3 \times 10^8 \text{ m/s})(1.6 \times 10^{-19} \text{ C})} \right), \\ &= \left(1.880 \times 10^{-7} \text{ W}\right)(2089682713.412) = 3930453.534 \text{ W} \approx 4 \text{ MW}. \end{aligned}$$

$$\boxed{P = 4 \text{ MW}}.$$

Now, we can compute the LHC version with protons. Because we weren't given the current this time, we will use the number of bunches and protons per bunch to compute the required power. We have that $I = \frac{Nve}{2\pi R}$, where N is the total number of particles, i.e. $N = (1240)(1.6 \times 10^{11}) = 1.984 \times 10^{14}$ protons. In the previous computation, we used $\frac{I}{e}$ to account for the full beam. Here, $\frac{I}{e} = \frac{Nc}{2\pi R}$. Combining this result with ΔU:

$$P = \Delta U \frac{I}{e} = P_{\text{rad}} \frac{2\pi R}{c}\frac{Nc}{2\pi R} = P_{\text{rad}} N.$$

Finally use the following set of data:

$$\begin{aligned} m_p &= .93827 \text{ GeV}, \quad E = 6800 \text{ GeV}, \\ R &= 4300 \text{ m} = 2.177 \times 10^{19}\text{GeV}^{-1}, \\ 1\text{GeV}^2 &= 2.43 \times 10^{14} \text{ W}, \quad \alpha = \frac{1}{137} \end{aligned}$$

and get the final result as:

$$\begin{aligned} P = &\left(\frac{2\,(6800 \text{ GeV})^4}{(137)3\,(0.93827 \text{ GeV})^4 \left(2.177 \times 10^{19}\text{GeV}^{-1}\right)^2} \frac{2.43 \times 10^{14} \text{ W}}{(1 \text{ GeV})^2} \right) (1.984 \times 10^{14}), \\ &= \left(6.883 \times 10^{-12} \text{ W}\right)(1.984 \times 10^{14}) = 1365.5872 \text{ W} \approx 1.4 \text{ kW}. \end{aligned}$$

$$\boxed{P = 1.4 \text{ kW}}.$$

c) Let's analyze the colliding beams case first. We can choose a representative from each beam (a single proton), and analyze the relativistic kinematics:

$$p_1^\mu = (E_1, \vec{p}_1) = (E, \vec{p}),$$
$$p_2^\mu = (E_2, \vec{p}_2) = (E, -\vec{p}).$$

Where the momentum is equal and opposite because they are going to collide with each other and we assume the same energy is given to each beam. Recall $E_i = m_i^2 + p_i^2$. Now, we compute $s = (p_1 + p_2)$:

$$\begin{aligned} s &= (p_1+p_2)^2 = ((E,\vec{p}) + (E,-\vec{p}))^2, \\ &= (2E)^2, \\ \therefore E &= \frac{\sqrt{s}}{2}. \end{aligned}$$

We want $\sqrt{s} = 14$ TeV, so the energy is just half of this quantity:

$$\boxed{E = 7 \text{ TeV}}.$$

Now, let's consider a target at rest and a single representative of the proton beam colliding with it:

$$p_1^\mu = (E_1, \vec{p}_1) = (E, \vec{p}),$$
$$p_2^\mu = (E_2, \vec{p}_2) = (m_p, 0).$$

Computing $\sqrt{s}$:

$$\begin{aligned} s &= (p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2, \\ &= p_1^2 + p_2^2 + 2(E_1E_2 - \vec{p}_1 \cdot \vec{p}_2), \\ &= 2m_p^2 + 2(Em_p - 0) = 2m_p^2 + 2Em_p, \\ \therefore E &= \frac{s - 2m_p^2}{2m_p} \approx \frac{s}{2m_p}. \end{aligned}$$

With $m_p = 0.000938$ TeV, we have

$$E = \frac{196 \text{ TeV}^2}{2(0.000938 \text{ TeV})}.$$

And finally:

$$\boxed{E = 104477.61 \text{ TeV}}.$$

We can see that because in the target version the invariant $\sqrt{s}$ scales with the square root of the energy, it requires more energy form the proton beam to achieve the same 14 TeV than in the two proton beam case.

Q2 Fermi Golden Rule

Questions:

Follow the class notes on time-dependent perturbation theory.

a) Drive the Fermi Golden rule for a time-dependent perturbation with an adiabatic switch-on:

$$V(t) = e^{\eta t}\overline{V},$$

for $\eta \to 0^+$.

b) Energy shift and Decay. Derive the energy shift and decay rate formulae

$$\Delta_R = V_{00} + \sum_{m\neq 0} \mathcal{P}\frac{|V_{mi}|^2}{E_0 - E_m},$$
$$\Gamma = -2\Delta_I = \frac{2\pi}{\hbar}\sum_{m\neq 0} |V_{mi}|^2 \delta(E_0 - E_m).$$

Explain the unitary condition.

Solutions

Notation. Following the class notes conventions, expressions such as $\delta_{E_n - E_0}$ or $c_{n\neq 0}$ are shorthand for $\delta(E_n - E_0)$ and c_n with $n \neq 0$, respectively. In the calculations below, such subscripts on δ or c_n indicate their arguments or conditions, not tensor indices.

a) Fermi Golden Rule.

We start with a perturbation on the Hamiltonian:

$$H \to H_0 + V(t)$$

Time dependence probing ($V(t)$) on the Hamiltonian.

$$H_0 |n\rangle = E_n |n\rangle,$$
$$|n(t)\rangle = e^{-iE_n t} |n\rangle.$$

We expand the eigenfunction as a linear combination of perturbed eigenstates.

$$|\psi(t))\rangle = \sum_n |n(t)\rangle \langle n(t)|\psi(t)\rangle$$
$$= \sum_n c_n(t) e^{-iE_n t} |n\rangle.$$

Using the Schrödinger equation for the new states:

$$\begin{aligned}
i\partial_t |\psi(t)\rangle &= (H_0 + V)|\psi(t)\rangle \\
\sum_n (E_n c_n + i\dot{c}_n) e^{-iE_n t}|n\rangle &= \sum_n (E_n + V) c_n e^{-iE_n t}|n\rangle \\
\sum_n i\dot{c}_n e^{-iE_n t}|n\rangle &= \sum_n V c_n e^{-iE_n t}|n\rangle \\
\therefore\ \dot{c}_n = -i \sum_m e^{iE_{nm}t} V_{nm} c_m & \\
E_{nm} = E_n - E_m & \\
V_{nm} = \langle n|V|m\rangle\,. &
\end{aligned}$$

In vector notation, $c_n \to \vec{c}$:

$$\begin{aligned}
\dot{\vec{c}} &= -iV_I \cdot \vec{c} \\
V_I &= e^{iH_0 t} V e^{-iH_0 t} \\
(V_I)_{nm} &= e^{iE_{mn}t} V_{nm} = \left\langle n \middle| e^{iH_0 t} V e^{-iH_0 t} \middle| m \right\rangle
\end{aligned}$$

The interaction picture emerges naturally, where V_I is the interaction version of V as a matrix. But how is the equation solved?

$$\begin{aligned}
\vec{c} &= e^{-i\int^t \mathrm{d}t' V_I}\vec{c}_0 \\
\vec{c} &\to T e^{-i\int^t \mathrm{d}t' V_I}\vec{c}_0
\end{aligned}$$

but $[V_I(t), V_I(t')] \neq 0$. Time ordering is necessary:

$$\vec{c} \to T e^{-i\int^{t'} \mathrm{d}t' V_I(t')}\vec{c}_0$$

Going back to the differential equation:

$$c_n(t) = c_n(0) - i\int^t \mathrm{d}t'\,(V_I)_{nm}(t') c_m(t')$$

$c_m(t')$ is the full solution. WE can iterate order by order. Let us fix the Initial conditions:

$$\begin{aligned}
& c_n(0) = \delta_{n0} \text{ Ground State} \\
& c_n(t) = \delta_{n0} - i\int^t \mathrm{d}t'\,(V_I)_{n0} + (-i)^2 \int^t \mathrm{d}t'\,(V_I)_{nm}(t') \int^{t'} \mathrm{d}t''\,(V_I)_{m0}(t'') + \dots \\
& c_n(t) = \delta_{n0} - i\int^t \mathrm{d}t'\,(V_I)_{n0} + \frac{1}{2!}(-i)^2 \int^t \mathrm{d}t'\,(V_I)_{nm}(t') \int^{t'} \mathrm{d}t''\,(V_I)_{m0}(t'') + \dots
\end{aligned}$$

We can change the integration limits of the last integral by noting that the integrated region is a triangle in the $t'' - t$ plane. As t'' cannot go further than the value of t', because t' itself is bounded by t. In figure 1, we see that the triangle's up-most point

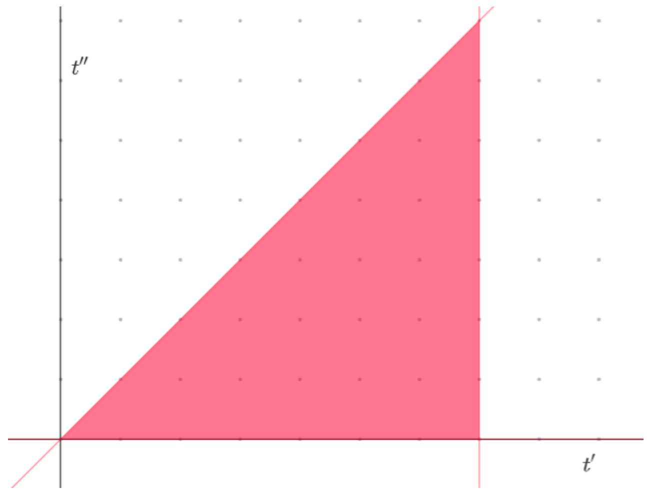

Figure 1: Integration Region.

is $(t', t'') = (t, t')$. But if we let the integral region cover the whole rectangle, we can change the limits of both variables from 0 to t. And divide by 2, as the square is double the original integration region. This is illustrated in figure 2. Before, we had a clear picture where $t'' < t'$, it was naturally ordered. But now that we have changed the integration limits, we must impose the ordering ourselves:

$$\int^t \mathrm{d}t'\, V_I(t') \int^{t'} \mathrm{d}t''\, V_I(t'') = \frac{1}{2!} \int^t \mathrm{d}t' \int^t \mathrm{d}t''\, T\{V_I(t')V_I(t'')\}$$

$$T\{e^{-i\int^t \mathrm{d}t' V_I(t')}\} = \sum_n \frac{(-i)^n}{n!} T\left\{\int^t \mathrm{d}t_1'\, V_I \int^t \mathrm{d}t_2' \dots\right\}$$

Here, T is the time order operator that imposes the ordering:

$$T\{A(t_1)A(t_2)\} = \begin{cases} A(t_1)A(t_2) & \text{if } t_1 > t_2 \\ A(t_2)A(t_1) & \text{if } t_2 > t_1. \end{cases}$$

Therefore we can write all terms in the following manner:

$$T\{e^{-i\int^t \mathrm{d}t' V_I(t')}\} \leftrightarrow \sum_n \frac{(-i)^n}{n!} T\left\{\int^t \mathrm{d}t_1'\, V_{I(t_1')} \int^t \mathrm{d}t_2'\, V_{I(t_2')} \dots\right\}.$$

The Fermi Golden Rule can be obtained already from leading order correction

$$c_n(t) \approx \delta_{n0} - i \int^t \mathrm{d}t'\, (V_I)_{n0}.$$

We start with

$$V(t) = \theta(t)\bar{V}.$$

Where $\theta(t)$ is the step-function, indicating where the perturbation started. In this case, $t = 0$. And $\overline{V}$ is the time-independent perturbation applied after switching on the perturbation.

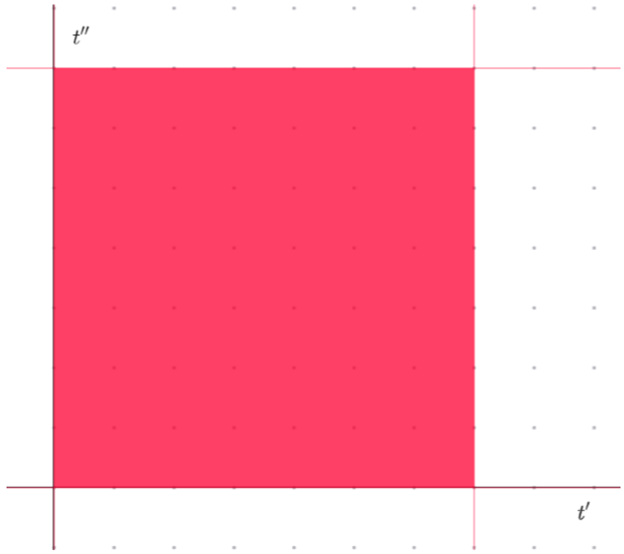

Figure 2: New integration Region.

If final state $n \neq 0$:

$$\begin{aligned} c_{n\neq 0}(t) &= -i\int_0^t \mathrm{d}t'\, \bar{V}_{n0}e^{-iE_{n0}t'} \\ &= -i\bar{V}_{n0}\frac{e^{iE_{n0}t}-1}{iE_{n0}}. \end{aligned}$$

Probability can be defined:

$$\begin{aligned} \therefore |c_{n\neq 0}|^2 &= |\bar{V}_{n0}|^2\frac{1}{E_{n0}^2}|e^{iE_{n0}t}-1|^2 \\ &= |\bar{V}_{n0}|^2\frac{4}{E_{n0}^2}\sin^2\left(\frac{E_{n0}t}{2}\right) \\ t\to\infty &= |\bar{V}_{n0}|^2[2\pi\delta_{E_n-E_0}]t \end{aligned}$$

Fermi Golden Rule:

$$\Gamma_{i\to f} = \text{ transition rate } = \frac{\mathrm{d}}{\mathrm{d}t}|c_f|^2 = |\bar{V}_{fi}|^2 2\pi\delta_{E_f-E_i}$$

$$\begin{aligned} \Gamma = \sum_f \Gamma_{i\to f} &= \sum_f |\bar{V}_{fi}|^2 2\pi\delta_{E_f-E_i} \\ &= \int \mathrm{d}E_f\, D_{E_f}|\bar{V}_{fi}|^2(2\pi)\delta_{E_f-E_i} \end{aligned}$$

Where D_{E_f} is the density of states. And it is integrated for a continuum of final states.

Now, let use an adiabatic switch-on $e^{\eta t}$ instead of a step function.

$$
\begin{aligned}
V(t) &\to e^{\eta t}\bar{V}, \quad t \to -\infty, \eta \to 0^+, V \to 0 \\
c_f^{(1)} &\to -i\bar{V}_{fi} \int_{-\infty}^{t} \mathrm{d}t'\, e^{i(E_f - E_i)t'} e^{\eta t'}, \bar{V}_{fi} = \langle f|\bar{V}|i\rangle \\
&= \frac{\bar{V}_{fi}}{E_i - E_f + i\eta} e^{-i(E_{fi} - i\eta)t} \\
c_f^0 &= \delta_{fi} \\
|c_f^{(1)}| &\leftrightarrow |\bar{V}_{fi}|^2 \frac{e^{2\eta t}}{(E_{if})^2 + \eta^2} \\
\Gamma_{i\to f} &= \frac{\mathrm{d}}{\mathrm{d}t} |c_f^{(1)}|^2 \text{ for large } t \\
&= |\bar{V}_{fi}|^2 \frac{2\eta}{E_{if}^2 + \eta^2} e^{2\eta t} \\
&\overset{\eta\to 0}{=} |\bar{V}_{fi}|^2 2\pi \delta_{E_i - E_f}.
\end{aligned}
$$

To understand the last two lines, consider the Principal Value P_V relation:

$$
\begin{aligned}
\frac{1}{x + i\delta} &= P_V\left(\frac{1}{x}\right) - i\pi\delta(x) \\
-2\cdot \mathrm{Im}\left(\frac{1}{x + i\delta}\right) = -2\cdot \frac{-\delta}{x^2 + \delta^2} &= \frac{2\delta}{x^2 + \delta^2} = 2\pi\delta(x)
\end{aligned}
$$

as $\delta \to 0$. The same Golden Rule is derived without trouble by using the previous results and considering a continuum of final states:

$$
\boxed{\Gamma = \int \mathrm{d}E_f\, D_{E_f} |\bar{V}_{fi}|^2 (2\pi) \delta_{E_f - E_i}}.
$$

b) Start by considering second order Effects:

$$
\begin{aligned}
c_f^{(0)} &= \delta_{fi} \\
c_f^{(1)} &= \bar{V}_{fi} \frac{1}{E_{fi} + i\eta} e^{i\left(E_{fi} - i\eta\right)t} \\
c_f^{(2)} &= \sum_{n'} \frac{\bar{V}_{fn'}\bar{V}_{n'i}}{(E_{fi} - 2i\eta)(E_{in'} + i\eta)} e^{i(E_{fi} - 2i\eta)t}
\end{aligned}
$$

Analyze when final state goes to initial state, in terms of energy.

$$
\begin{aligned}
f &\approx i \\
C_{f\to i} &\approx 1 - i\bar{V}_{ii} \frac{1}{\eta} e^{\eta t} + \\
(-i)^2 |\bar{V}_{ii}|^2 &\frac{1}{2\eta^2} e^{2\eta t} + \\
-i \sum_{n'} |\bar{V}_{in'}|^2 &\frac{1}{E_{in'} + i\eta} \frac{1}{2\eta} e^{2\eta t}
\end{aligned}
$$

Where $\sum_{n'}$ means that $n' \neq i$. When $\eta \to 0$ then $c_{f\approx i} \to \infty$. As $\eta \to 0$, $c_{f\to i}$ diverges. To find a way out, take the time derivative:

$$\dot{c}_{f\to i} = -i\bar{V}_{ii}e^{\eta t} + (-i)^2|\bar{V}_{ii}|^2\frac{1}{\eta}e^{2\eta t} - i\sum_{n'}|\bar{V}_{in'}|^2\frac{1}{E_{in'}+i\eta}e^{2\eta t}$$

Still diverges as $\eta \to 0$, so now take the ratio $\frac{\dot{c}_{f\to i}}{c_{f\to i}}$:

$$\begin{aligned}\frac{\dot{c}_{f\to i}}{c_{f\to i}} &= \frac{-i\bar{V}_{ii}e^{\eta t} + (-i)^2|\bar{V}_{ii}|^2\frac{1}{\eta}\dot{e^{2\eta t}} - i\sum_{n'}|\bar{V}_{in'}|^2\frac{1}{E_{in'}+i\eta}e^{2\eta t}}{1 - i\bar{V}_{ii}\frac{1}{\eta}e^{\eta t} + \dots} \\ &\approx -i\bar{V}_{ii} - i\sum_{n'}|\bar{V}_{n'i}|^2\frac{1}{E_{in'}+i\eta}\end{aligned}$$

Where we have used the following argument for small x:

$$(x+x^2)\frac{1}{1+x} \approx (1-x+\dots)(x+x^2) \approx x + x^2 - x^2 + x^3 \approx x - x^3$$

Now is safe to take $\eta \to 0$, all divergence terms cancel. This means that $\frac{\mathrm{d}}{\mathrm{d}t}\ln(c_{f\to i})$ is finite, therefore $c^R_{f\to i}$ is proportional to $e^{-i\Delta t}$.

$$-i\Delta = \frac{\mathrm{d}}{\mathrm{d}t}\ln(c_{f\to i})$$

$$\Delta = \bar{V}_{ii} + \underbrace{\sum_{n'}|\bar{V}_{in'}|^2\frac{1}{E_i - E_{n'} + i\eta}}_{\text{Complex quantity}} = \Delta_R + i\Delta_I$$

$$\boxed{\begin{aligned}\Delta_R &= \bar{V}_{ii} + \sum_{n'}|\bar{V}_{in'}|^2\,P\left(\frac{1}{E_i - E_{n'}}\right), \\ -2\Delta_I &= 2\pi\sum_{n'}|\bar{V}_{in'}|^2\delta_{E_i-E_{n'}} = \Gamma.\end{aligned}}$$

Where Γ is the width, as dictated by Fermi Golden Rule. And Δ_R is the energy shift.

$$|c^R_{f\approx i}|^2 = |e^{-i\Delta t}|^2 = e^{2\Delta_I t} \approx 1 + 2\Delta_I t$$

This is the probability of $i \to f \approx i$:

$$\Gamma t = \sum_{f=n'\neq i}\Gamma_{i\to n'\neq i} = \sum_{n}2\pi|\bar{V}_{in'}|^2\delta_{E_i-E_f}t$$

And this is the probability of $i \to f = n'$ not equal to i.

$$|c^R_{f\to i}|^2 + \sum_{n'}\Gamma_{i\to n'}t = 1.$$

The probability of not having n' as a final state plus the probability of having i as the final state should add up to 1. And unitary operators will preserve the probability. Unitarity of time evolution requires that

$$\sum_f |c_f(t)|^2 = 1.$$

The imaginary part of the energy shift controls how much probability amplitude is "lost". As in, we have a certain probability of staying in the same state ($f \to i$), but as we say, some of this energy shift is imaginary and therefore introduces a negative exponential decay. This leak is picked up by the probability of moving to different final states which should match the sum of transition rates into all other possible states. Exactly what the Fermi Golden Rules is talking about. Therefore, the unitary condition requires exactly that $\Gamma = -2\Delta_I$.

Q3 Baker-Campbell-Hausdorff Formula

Questions:

a) Show that

$$e^{-xA}Be^{xA} = \sum_{n=0}^{\infty} \frac{x^n}{n!} C\left[B; A, n\right]$$

with

$$\begin{aligned} C\left[B; A, n\right] &= \left[C\left[B; A, n-1\right], A\right], \\ C\left[B; A, 0\right] &= B. \end{aligned}$$

b) Given that

$$Z(x) = e^{W(x)},$$

prove that

$$Z^{-1}\frac{\mathrm{d}Z}{\mathrm{d}x} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} C\left[\frac{\mathrm{d}W}{\mathrm{d}x}; W, n\right].$$

c) Apply the result in b) with

$$\begin{aligned} Z(x) &= e^{xA}e^{xB}, \\ W(x) &= \sum_{n=1} x^n W_n. \end{aligned}$$

Show the leading order results:

$$\begin{aligned} W_1 &= A + B, \\ W_2 &= \frac{1}{2}[A, B], \\ W_3 &= \frac{1}{12}[[A, B], (B - A)]. \end{aligned}$$

Solutions

a) To show the formula, we expand each exponential:

$$e^{xA} = \sum_{n=0}^{\infty} \frac{x^n}{n!} A^n, \quad e^{-xA} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} A^n.$$

And we substitute into the left-hand side:

$$\begin{aligned}
e^{-xA}Be^{xA} &= \left(1 - xA + \frac{x^2}{2!}A^2 - \frac{x^3}{3!}A^3 + \dots\right) B \left(1 + xA + \frac{x^2}{2!}A^2 + \frac{x^3}{3!}A^3 \dots\right), \\
&= B + xBA + \frac{x^2}{2!}BA^2 - xAB - x^2ABA - \frac{x^3}{2!}ABA^2 \\
&+ \frac{x^2}{2!}A^2B + \frac{x^3}{2!}A^2BA + \frac{x^4}{4!}A^2BA^2 - \frac{x^3}{3!}A^3B + \frac{x^3}{3!}BA^3 \dots, \\
&= B + x\left(BA - AB\right) + \frac{x^2}{2!}\left(BA^2 - 2ABA + A^2B\right) \\
&+ \frac{x^3}{3!}(BA^3 - 3ABA^2 + 3A^2BA - A^3B) + \dots.
\end{aligned}$$

Let's analyze each power of x separately:

- x^0: This is just B.
- x^1:

$$BA - AB = [B, A].$$

- x^2:

$$\begin{aligned}
BA^2 - 2ABA + A^2B &= BAA - ABA - ABA + AAB, \\
&= (BA - AB)A - A(BA - AB), \\
&= [A, B]A - A[A, B], \\
&= [[A, B], A].
\end{aligned}$$

- x^3:

$$\begin{aligned}
&BA^3 - 3ABA^2 + 3A^2BA - A^3B = \\
&= BAAA - ABAA - ABAA - ABAA + AABA + AABA + AABA - AAAB, \\
&= BAAA - ABAA - ABAA + AABA - ABAA + AABA + AABA - AAAB, \\
&= (BA - AB)AA - A(AB - BA)A - A(BA - AB)A - AA(AB - BA), \\
&= [B, A]AA - A[B, A]A - A[B, A]A - AA[B, A], \\
&= ([B, A]A - A[B, A])A - A([B, A]A - A[B, A]), \\
&= [[B, A], A]A - A[[B, A], A], \\
&= [[[B, A], A], A].
\end{aligned}$$

We can see that we can build the operators on each power of x recursively. As the solution suggests, we can define $C[B; A, 0] = B$ and build recursively:

$$\begin{aligned}
C[B; A, 0] &= B, \\
C[B; A, 1] &= [C[B; A, 0], A] = [B, A], \\
C[B; A, 2] &= [C[B; A, 1], A] = [[B, A], A], \\
C[B; A, 3] &= [C[B; A, 2], A] = [[[B, A], A], A], \\
&\vdots \\
C[B; A, n] &= [C[B; A, n - 1], A].
\end{aligned}$$

Therefore, the result is:

$$\boxed{e^{-xA}Be^{xA} = \sum_{n=0}^{\infty} \frac{x^n}{n!} C[B;A,n]}.$$

b) To prove the relation, we identify $Z^{-1} = e^{-W(x)}$ and expand it and Z into Taylor series as in part a):

$$\begin{aligned}
&Z^{-1}\frac{\mathrm{d}Z}{\mathrm{d}x} = \\
&= \left(1 - W + \frac{1}{2!}W^2 - \frac{1}{3!}W^3 + \dots\right) \frac{\mathrm{d}}{\mathrm{d}x}\left(1 + W + \frac{1}{2!}W^2 + \frac{1}{3!}W^3 + \dots\right), \\
&= \left(1 - W + \frac{1}{2!}W^2 - \frac{1}{3!}W^3 + \dots\right) \times \\
&\times \left(\frac{\mathrm{d}W}{\mathrm{d}x} + \frac{1}{2!}\left(\frac{\mathrm{d}W}{\mathrm{d}x}W + W\frac{\mathrm{d}W}{\mathrm{d}x}\right) + \frac{1}{3!}\left(\frac{\mathrm{d}W}{\mathrm{d}x}W^2 + W\frac{\mathrm{d}W}{\mathrm{d}x}W + W^2\frac{\mathrm{d}W}{\mathrm{d}x}\right) + \dots\right), \\
&= W + \frac{1}{2!}\left(\frac{\mathrm{d}W}{\mathrm{d}x}W + W\frac{\mathrm{d}W}{\mathrm{d}x}\right) + \frac{1}{3!}\left(\frac{\mathrm{d}W}{\mathrm{d}x}W^2 + W\frac{\mathrm{d}W}{\mathrm{d}x}W + W^2\frac{\mathrm{d}W}{\mathrm{d}x}\right) + \\
&- W\frac{\mathrm{d}W}{\mathrm{d}x} - \frac{1}{2!}\left(W\frac{\mathrm{d}W}{\mathrm{d}x}W + W^2\frac{\mathrm{d}W}{\mathrm{d}x}\right) + \frac{1}{2!}W^2\frac{\mathrm{d}W}{\mathrm{d}x} + \dots, \\
&= \frac{\mathrm{d}W}{\mathrm{d}x} + \frac{1}{2!}\left(\frac{\mathrm{d}W}{\mathrm{d}x} + W\frac{\mathrm{d}W}{\mathrm{d}x} - 2W\frac{\mathrm{d}W}{\mathrm{d}x}\right) \\
&+ \frac{1}{3!}\left(\frac{\mathrm{d}W}{\mathrm{d}x}W^2 + W\frac{\mathrm{d}W}{\mathrm{d}x}W + W^2\frac{\mathrm{d}W}{\mathrm{d}x} - 3W\frac{\mathrm{d}W}{\mathrm{d}x}W - 3W^2\frac{\mathrm{d}W}{\mathrm{d}x} + 3W^2\frac{\mathrm{d}W}{\mathrm{d}x}\right) + \dots \\
&= \frac{\mathrm{d}W}{\mathrm{d}x} + \frac{1}{2!}\left(\frac{\mathrm{d}W}{\mathrm{d}x}W - W\frac{\mathrm{d}W}{\mathrm{d}x}\right) + \frac{1}{3!}\left(\frac{\mathrm{d}W}{\mathrm{d}x}W^2 - 2W\frac{\mathrm{d}W}{\mathrm{d}x}W + W^2\frac{\mathrm{d}W}{\mathrm{d}x}\right) + \dots.
\end{aligned}$$

As before, we can analyze each group of operators, but now they have an inverse factorial:

- $\frac{1}{1!}$: This is just $\frac{\mathrm{d}W}{\mathrm{d}x}$.
- $\frac{1}{2!}$:

$$\frac{\mathrm{d}W}{\mathrm{d}x}W - W\frac{\mathrm{d}W}{\mathrm{d}x} = \left[\frac{\mathrm{d}W}{\mathrm{d}x}, W\right].$$

- $\frac{1}{3!}$:

$$
\begin{aligned}
&\frac{\mathrm{d}W}{\mathrm{d}x}W^2 - 2W\frac{\mathrm{d}W}{\mathrm{d}x}W + W^2\frac{\mathrm{d}W}{\mathrm{d}x} = \\
&\qquad = \frac{\mathrm{d}W}{\mathrm{d}x}WW - W\frac{\mathrm{d}W}{\mathrm{d}x}W - W\frac{\mathrm{d}W}{\mathrm{d}x}W + WW\frac{\mathrm{d}W}{\mathrm{d}x}, \\
&\qquad = \left(\frac{\mathrm{d}W}{\mathrm{d}x}W - W\frac{\mathrm{d}W}{\mathrm{d}x}\right)W - W\left(\frac{\mathrm{d}W}{\mathrm{d}x}W - W\frac{\mathrm{d}W}{\mathrm{d}x}\right), \\
&\qquad = \left[\frac{\mathrm{d}W}{\mathrm{d}x}, W\right]W - W\left[\frac{\mathrm{d}W}{\mathrm{d}x}, W\right], \\
&\qquad = \left[\left[\frac{\mathrm{d}W}{\mathrm{d}x}, W\right], W\right].
\end{aligned}
$$

As before, we can deduce a pattern recursively:

$$
\begin{aligned}
C\left[\frac{\mathrm{d}W}{\mathrm{d}x}; W, 0\right] &= \frac{\mathrm{d}W}{\mathrm{d}x}, \\
C\left[\frac{\mathrm{d}W}{\mathrm{d}x}; W, 1\right] &= \left[C\left[\frac{\mathrm{d}W}{\mathrm{d}x}; W, 0\right], W\right] = \left[\frac{\mathrm{d}W}{\mathrm{d}x}, W\right], \\
C\left[\frac{\mathrm{d}W}{\mathrm{d}x}; W, 2\right] &= \left[C\left[\frac{\mathrm{d}W}{\mathrm{d}x}; W, 1\right], W\right] = \left[\left[\frac{\mathrm{d}W}{\mathrm{d}x}, W\right], W\right], \\
&\vdots \\
C\left[\frac{\mathrm{d}W}{\mathrm{d}x}; W, n\right] &= \left[C\left[\frac{\mathrm{d}W}{\mathrm{d}x}; W, n-1\right], W\right].
\end{aligned}
$$

But, we see that the factorial has shifted. The first term is $\frac{1}{1!}$, the second $\frac{1}{2!}$; instead of starting at $\frac{1}{0!}$. Therefore we can write the result as:

$$
\boxed{Z^{-1}\frac{\mathrm{d}Z}{\mathrm{d}x} = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}C\left[\frac{\mathrm{d}W}{\mathrm{d}x}; W, n\right]}.
$$

c) Let's start by substituting $Z(x)$ in the left-hand side, and $W(x)$ in the right-hand side:

- RHS:

$$
\begin{aligned}
Z^{-1}\frac{\mathrm{d}Z}{\mathrm{d}x} &= e^{-xB}e^{-xA}\left(\frac{\mathrm{d}e^{xA}}{\mathrm{d}x}e^{xB} + e^{xA}\frac{\mathrm{d}e^{xB}}{\mathrm{d}x}\right), \\
&= e^{-xB}e^{-xA}\left(e^{xA}Ax^{xB} + e^{xA}Be^{xB}\right), \\
&= e^{-xB}Ae^{xB} + e^{-xB}Be^{xB} = e^{-xB}Ae^{xB} + B, \\
&= \sum_{n=0}^{\infty}\frac{x^n}{n!}C\left[A; B, n\right] + B, \\
&= A + B + x[A, B] + \frac{x^2}{2!}[[A, B], B] + \ldots
\end{aligned}
$$

- LHS: If $W(x) = \sum_{n=1}^{\infty} x^n W_n$, then
$$\frac{\mathrm{d}W}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=1}^{\infty} x^n W_n\right) = \sum_{n=1}^{\infty} n x^{n-1} W_n.$$

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(n+1)!} C\left[\frac{\mathrm{d}W}{\mathrm{d}x}; W, n\right] &= \frac{\mathrm{d}W}{\mathrm{d}x} + \frac{1}{2!}\left[\frac{\mathrm{d}W}{\mathrm{d}x}, W\right] + \dots, \\
&= W_1 + 2xW_2 + 3x^2W_3 + \dots \\
&+ \frac{1}{2}\left(x[W_1, W_1] + x^2[W_1, W_2] + 2x^2[W_2, W_1] \dots\right), \\
&= W_1 + 2xW_2 + 3x^2W_3 + \dots \\
&+ \frac{1}{2}\left(-x^2[W_2, W_1] + 2x^2[W_2, W_1] \dots\right), \\
&= W_1 + 2xW_2 + 3x^2W_3 + \frac{x^2}{2}[W_2, W_1] + \dots, \\
&= W_1 + x(2W_2) + x^2\left(3W_3 + \frac{1}{2}[W_2, W_1]\right) + \dots
\end{aligned}$$

We match LHS with RHS by grouping powers of x:

- x^0:
$$A + B = W_1.$$
- x^1:
$$[A, B] = 2W_2, \qquad \therefore W_2 = \frac{1}{2}[A, B].$$
- x^2:
$$\begin{aligned}
\frac{1}{2}[[A, B], B] &= 3W_3 + \frac{1}{2}[W_2, W_1], \\
\frac{1}{2}[[A, B], B] &= 3W_3 + \frac{1}{4}[[A, B], A + B], \\
\frac{1}{2}[[A, B], B] &= 3W_3 + \frac{1}{4}[[A, B], A] + \frac{1}{4}[[A, B], B], \\
\frac{1}{4}[[A, B], B] &= 3W_3 + \frac{1}{4}[[A, B], A], \\
\frac{1}{4}([[A, B], B] - [[A, B], A]) &= 3W_3, \\
\frac{1}{12}[[A, B], (B - A)] &= W_3.
\end{aligned}$$

Therefore, the leading order results are:

$$\boxed{\begin{aligned}
&W_1 = A + B, \\
&W_2 = \frac{1}{2}[A, B], \\
&W_3 = \frac{1}{12}[[A, B], (B - A)].
\end{aligned}}$$

Q4 Bogoliubov Transform and Quasiparticles

Questions:

a) Starting with the bosonic operators satisfying

$$[a, a^\dagger] = 1.$$

Consider the generator Q and the similarity transformation operator $U(\theta)$

$$Q = \frac{1}{2}\left(a^2 - \left(a^\dagger\right)^2\right), \quad U(\theta) = e^{\theta Q}.$$

Derive the following:

$$\begin{aligned} A(\theta) &= U^{-1} a U = a\cosh(\theta) - a^\dagger \sinh(\theta), \\ A(\theta)^\dagger &= U^{-1} a^\dagger U = a^\dagger \cosh(\theta) - a \sinh(\theta), \end{aligned}$$

and that

$$[A(\theta), A^\dagger(\theta)] = 1.$$

b) Show that the RPA vacuum

$$|0'(\theta)\rangle = U^{-1}(\theta)\,|0\rangle\,,$$

is annihilated by $A(\theta)$:

$$A(\theta)\,|0'(\theta)\rangle = 0.$$

c) Consider the quadratic bosonic Hamiltonian

$$H = \omega a^\dagger a + \frac{\Delta}{2}\left(a^2 + \left(a^\dagger\right)^2\right).$$

Compute the RPA vacuum energy

$$E_0(\theta) = \langle 0'(\theta)|H|0'(\theta)\rangle\,.$$

Treat θ as a variational parameter and find the stationary condition for $\theta = \overline{\theta}$

$$\frac{\partial E_0(\theta)}{\partial \theta} = 0,$$

and show that

$$\overline{\theta} = -\frac{1}{2}\tanh^{-1}\left(\frac{\Delta}{\omega}\right)$$

Plug this back into the Hamiltonian to show that

$$H = \Omega A^\dagger A + \overline{E},$$

where

$$\Omega = \sqrt{\omega^2 - \Delta^2}, \quad \overline{E} = -\frac{\omega - \Omega}{2}.$$

The interaction term is "transformed" away. Discuss what happens when $\omega^2 < \Delta^2$.

Solutions

a) Using the formula derived in Question 3 part a), we can compute $A(\theta)$ and $A(\theta)^\dagger$:

$$\begin{aligned} A(\theta) &= U^{-1}aU = e^{-\theta Q}ae^{\theta Q}, \\ &= \sum_{n=0}^{\infty}\frac{\theta^n}{n!}C[a;Q,n]. \end{aligned}$$

So we are left computing $C[a;Q,n]$ for some n, and find a pattern.

$$\begin{aligned} C[a;Q,0] &= a, \\ C[a;Q,1] &= [C[a;Q,0],Q] = [a,Q] = \frac{1}{2}\left(\left[a,a^2\right] - \left[a,(a^\dagger)^2\right]\right), \\ &= \frac{1}{2}\left([a,a]a + a[a,a] - [a,a^\dagger]a^\dagger - a^\dagger[a,a^\dagger]\right), \\ &= \frac{1}{2}(-a^\dagger - a^\dagger) = -a^\dagger, \\ C[a;Q,2] &= [C[a;Q,2],Q] = [-a^\dagger,Q] = -\frac{1}{2}\left(\left[a^\dagger,a^2\right] - \left[a^\dagger,(a^\dagger)^2\right]\right), \\ &= -\frac{1}{2}\left([a^\dagger,a]a + a[a^\dagger,a] - [a^\dagger,a^\dagger]a^\dagger - a^\dagger[a^\dagger,a^\dagger]\right), \\ &= -\frac{1}{2}(-a-a) = a, \\ C[a;Q,3] &= [C[a;Q,2],Q] = [a,Q] = -a^\dagger, \\ C[a;Q,4] &= [C[a;Q,3],Q] = [-a^\dagger,Q] = a. \end{aligned}$$

Where we have used that $[a,a^\dagger]=1$ and $[a,a]=[a^\dagger,a^\dagger]=0$. Plugging this back into the previous result:

$$A(\theta) = a - \frac{\theta}{1!}a^\dagger + \frac{\theta^2}{2!}a - \frac{\theta^3}{3!}a^\dagger + \frac{\theta^4}{4!}a \ldots.$$

We can see that the infinite sum splits into even and odd powers of θ, and that on each one, we can factor the operator:

$$A(\theta) = a\left(1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\ldots\right) - a^\dagger\left(\frac{\theta}{1!} + \frac{\theta^3}{3!}\ldots\right).$$

We identify the Taylor Series for $\cosh(\theta)$ and $\sinh(\theta)$:

$$\cosh(\theta) = \sum_0^\infty \frac{\theta^{2k}}{(2k)!}, \quad \sinh(\theta) = \sum_0^\infty \frac{\theta^{1+2k}}{(1+2k)!}.$$

$$\boxed{A(\theta) = a\cosh(\theta) - a^\dagger\sinh(\theta)}.$$

We can follow a similar procedure to find $A(\theta)^\dagger$, or just apply the Hermitian adjoint to the last result:

$$A(\theta)^\dagger = (a\cosh(\theta) - a^\dagger \sinh(\theta))^\dagger,$$

leading to

$$\boxed{A(\theta)^\dagger = a^\dagger \cosh(\theta) - a\sinh(\theta)}.$$

Now, compute the commutator between $A(\theta)$ and its Hermitian Conjugate:

$$\begin{aligned}
[A(\theta), A(\theta)^\dagger] &= [a\cosh(\theta) - a^\dagger \sinh(\theta), a^\dagger \cosh(\theta) - a\sinh(\theta)], \\
&= \cosh^2(\theta)[a, a^\dagger] - \cosh(\theta)\sinh(\theta)[a, a], \\
&- \sinh(\theta)\cosh(\theta)[a^\dagger, a^\dagger] + \sinh^2(\theta)[a^\dagger, a], \\
&= \cosh^2(\theta) - \sinh^2(\theta).
\end{aligned}$$

Where we have used the commutations relations of a and $a^\dagger$ as before, and we can use the well know hyperbolic relation to conclude that

$$\boxed{[A(\theta), A(\theta)^\dagger] = 1}.$$

b) To solve this problem, let's write $A(\theta)$ as used in part a) and apply it to the RPA vacuum:

$$\begin{aligned}
A(\theta)\left|0'(\theta)\right\rangle &= \left(U^{-1}aU\right)\left(U^{-1}\left|0\right\rangle\right), \\
&= U^{-1}aUU^{-1}\left|0\right\rangle, \\
&= U^{-1}a\left|0\right\rangle.
\end{aligned}$$

Annihilation operators acting on the vacuum map it to zero, therefore

$$\boxed{A(\theta)\left|0'(\theta)\right\rangle = 0}. \tag{1}$$

c) Let's compute directly the RPA vacuum energy:

$$\begin{aligned}
E_0(\theta) &= \left\langle 0'(\theta)\middle|H\middle|0'(\theta)\right\rangle, \\
&= \left\langle 0\middle|UHU^{-1}\middle|0\right\rangle, \\
&= \left\langle 0\middle|\omega Ua^\dagger aU^{-1} + \frac{\Delta}{2}Ua^2U^{-1} + \frac{\Delta}{2}U(a^\dagger)^2U^{-1}\middle|0\right\rangle, \\
&= \left\langle 0\middle|\omega\left[Ua^\dagger U^{-1}\right]UaU^{-1} + \frac{\Delta}{2}[UaU^{-1}]UaU^{-1} + \frac{\Delta}{2}[Ua^\dagger U^{-1}]Ua^\dagger U^{-1}\middle|0\right\rangle.
\end{aligned}$$

Where we have inserted $I = U^{-1}U$ in the middle of the $a/a^\dagger$ operators. Now, notice the following:

$$\begin{aligned}
A(\theta) &= U^{-1}aU = e^{-\theta Q}ae^{\theta Q}, \\
\text{Let } \theta &\to -\theta, \\
A(-\theta) &= e^{\theta Q}ae^{-\theta Q}, \\
\therefore A(-\theta) &= UaU^{-1}.
\end{aligned}$$

And apply it to the previous line of results:

$$E_0(\theta) = \left\langle 0 \middle| \omega A(-\theta)^\dagger A(-\theta) + \frac{\Delta}{2} A^2(-\theta) + \frac{\Delta}{2} (A(-\theta)^\dagger)^2 \middle| 0 \right\rangle .$$

Now, let's analyze the effect of $A(-\theta)$ and $A(-\theta)^\dagger$ to the vacuum:

$$\begin{aligned}
A(-\theta) \left|0\right\rangle &= \left(\cosh(-\theta) a - \sinh(-\theta) a^\dagger\right) \left|0\right\rangle , \\
&= -\sinh(-\theta) a^\dagger \left|0\right\rangle , \\
&= \sinh(\theta) a^\dagger \left|0\right\rangle , \\
\left\langle 0\right| A(-\theta)^\dagger &= \left\langle 0\right| a \sinh(\theta), \\
A(-\theta)^\dagger \left|0\right\rangle &= \left(\cosh(-\theta) a^\dagger - \sinh(-\theta) a\right) \left|0\right\rangle , \\
&= \cosh(-\theta) a^\dagger \left|0\right\rangle , \\
&= \cosh(\theta) a^\dagger \left|0\right\rangle , \\
\left\langle 0\right| A(-\theta) &= \left\langle 0\right| a(\cosh(\theta)),
\end{aligned}$$

Where we used the fact that $\sinh(-x) = -\sinh(x)$ and $\cosh(-x) = \cosh(x)$. We continue:

$$\begin{aligned}
E_0(\theta) &= \left\langle 0 \middle| \omega \sinh^2(\theta) a a^\dagger + \Delta \sinh(\theta)\cosh(\theta) a a^\dagger \middle| 0 \right\rangle , \\
&= \left(\omega \sinh^2(\theta) + \Delta \sinh(\theta)\cosh(\theta)\right) \left\langle 0 \middle| a a^\dagger \middle| 0 \right\rangle , \\
&= \left(\omega \sinh^2(\theta) + \Delta \sinh(\theta)\cosh(\theta)\right) \left\langle 0 \middle| [a, a^\dagger] + a^\dagger a \middle| 0 \right\rangle , \\
&= \left(\omega \sinh^2(\theta) + \Delta \sinh(\theta)\cosh(\theta)\right) \left\langle 0 | 1 | 0 \right\rangle , \\
&= \omega \sinh^2(\theta) + \Delta \sinh(\theta)\cosh(\theta).
\end{aligned}$$

Where we have done $[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$ in the last couple lines. Now, take the derivative with respect θ and equate to zero:

$$\begin{aligned}
\frac{\partial E_0(\theta)}{\partial \theta} &= 2\omega \sinh(\theta)\cosh(\theta) + \Delta(\sinh(x)^2 + \cosh(x)^2), \\
&= \omega \sinh(2\theta) + \Delta \cosh(2\theta) \stackrel{!}{=} 0,
\end{aligned}$$

Where we used the double angle hyperbolic formulas:

$$\begin{aligned}
e^x &= \cosh(x) + \sinh(x), \\
e^{2x} &= \cosh(2x) + \sin(2x) = (\cosh(x) + \sinh(x))^2, \\
&= \cosh(x)^2 + \sinh(x)^2 + 2\sinh(x)\cosh(x).
\end{aligned}$$

By parity of the functions, $\cosh(2x) = \cosh(x)^2 + \sinh(x)^2$, as both sides are even functions, and $\sinh(2x) = 2\sinh(x)\cosh(x)$ as both sides are odd. We continue solving for $\overline{\theta}$:

$$\begin{aligned}
\omega \tanh\big(2\overline{\theta}\big) + \Delta &= 0,\\
\tanh\big(2\overline{\theta}\big) &= -\frac{\Delta}{\omega},\\
\overline{\theta} &= \frac{1}{2}\tanh^{-1}\left(-\frac{\Delta}{\omega}\right).
\end{aligned}$$

Finally, we get that

$$\boxed{\overline{\theta} = -\frac{1}{2}\tanh^{-1}\left(\frac{\Delta}{\omega}\right)}.$$

Using that $\tanh^{-1}(-x) = -\tanh^{-1}(x)$.

Lastly, we plug in $\overline{\theta}$ back in H. But to do so, we need to write $a/a^\dagger$ in terms of $A(\theta)/A^\dagger(\theta)$. We can write the relation as a matrix equation:

$$\begin{pmatrix} A \\ A^\dagger \end{pmatrix} = \begin{pmatrix} \cosh(\theta) & -\sinh(\theta) \\ -\sinh(\theta) & \cosh(\theta) \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}.$$

Now, we find the inverse of the matrix, and find the inverse relations. Notice that the determinant of the matrix is 1.

$$\begin{pmatrix} a \\ a^\dagger \end{pmatrix} = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix} \begin{pmatrix} A \\ A^\dagger \end{pmatrix}.$$

We substitute in the Hamiltonian, and evaluate at $\overline{\theta}$. We will use $c \equiv \cosh(\theta)$ and $s \equiv \sinh(\theta)$:

$$\begin{aligned}
H &= \omega a^\dagger a + \frac{\Delta}{2}(a^2 + (a^\dagger)^2),\\
&= \omega(sA + cA^\dagger)(cA + sA^\dagger) + \frac{\Delta}{2}((cA + sA^\dagger)^2 + (sA + cA^\dagger)^2),\\
&= \omega(sc(A^2 + (A^\dagger)^2) + s^2 AA^\dagger + c^2 A^\dagger A) + \frac{\Delta}{2}(c^2A^2 + s^2(A^\dagger)^2 + cs(AA^\dagger + A^\dagger A)) +\\
&+ \frac{\Delta}{2}(s^2A^2 + c^2(A^\dagger)^2 + cs(AA^\dagger + A^\dagger A)),\\
&= \omega(sc(A^2 + (A^\dagger)^2) + s^2 AA^\dagger + c^2 A^\dagger A)\\
&+ \frac{\Delta}{2}((c^2 + s^2)(A^2 + (A^\dagger)^2) + 2cs(AA^\dagger + A^\dagger A))
\end{aligned}$$

Analyze the coefficients, here we will evaluate at $\overline{\theta}$:

- $A^2 + (A^\dagger)^2$:

$$\begin{aligned}
\omega sc + \frac{\Delta}{2}(c^2 + s^2) &= \frac{\omega}{2}\sinh\big(2\overline{\theta}\big) + \frac{\Delta}{2}\cosh\big(2\overline{\theta}\big),\\
&= \frac{\omega}{2}\sinh\left(-\tanh^{-1}\left(\frac{\Delta}{\omega}\right)\right) + \frac{\Delta}{2}\cosh\left(-\tanh^{-1}\left(\frac{\Delta}{\omega}\right)\right),\\
&= -\frac{\omega}{2}\frac{\Delta}{\sqrt{\omega^2 - \Delta^2}} + \frac{\Delta}{2}\frac{\omega}{\sqrt{\omega^2 - \Delta^2}} = 0.
\end{aligned}$$

- $A^\dagger A$:

$$\begin{aligned}
\Delta cs + \omega c^2 &= \frac{\Delta}{2}\sinh\left(2\overline{\theta}\right) + \frac{\omega}{2}(\cosh\left(2\overline{\theta}\right) + 1),\\
&= -\frac{\Delta}{2}\frac{\Delta}{\sqrt{\omega^2-\Delta^2}} + \frac{\omega}{2}\frac{\omega}{\sqrt{\omega^2-\Delta^2}} + \frac{\omega}{2},\\
&= \frac{\omega^2-\Delta^2}{2\sqrt{\omega^2-\Delta^2}} + \frac{\omega}{2},\\
&= \frac{\sqrt{\omega^2-\Delta^2}}{2} + \frac{\omega}{2}.
\end{aligned}$$

- $AA^\dagger$:

$$\begin{aligned}
\Delta cs + \omega s^2 &= \frac{\Delta}{2}\sinh\left(2\overline{\theta}\right) + \frac{\omega}{2}(\cosh\left(2\overline{\theta}\right) - 1),\\
&= -\frac{\Delta}{2}\frac{\Delta}{\sqrt{\omega^2-\Delta^2}} + \frac{\omega}{2}\frac{\omega}{\sqrt{\omega^2-\Delta^2}} - \frac{\omega}{2},\\
&= \frac{\sqrt{\omega^2-\Delta^2}}{2} - \frac{\omega}{2}.
\end{aligned}$$

Substituting back:

$$\begin{aligned}
H &= \left(\frac{\sqrt{\omega^2-\Delta^2}}{2} - \frac{\omega}{2}\right)AA^\dagger + \left(\frac{\sqrt{\omega^2-\Delta^2}}{2} + \frac{\omega}{2}\right)A^\dagger A,\\
&= \left(\frac{\sqrt{\omega^2-\Delta^2}}{2} - \frac{\omega}{2}\right)\left(A^\dagger A + [A, A^\dagger]\right) + \left(\frac{\sqrt{\omega^2-\Delta^2}}{2} + \frac{\omega}{2}\right)A^\dagger A,\\
&= \sqrt{\omega^2-\Delta^2}A^\dagger A + \left(\frac{\sqrt{\omega^2-\Delta^2}}{2} - \frac{\omega}{2}\right),\\
&= \sqrt{\omega^2-\Delta^2}A^\dagger A - \frac{\omega-\sqrt{\omega^2-\Delta^2}}{2}.
\end{aligned}$$

Now define $\Omega = \sqrt{\omega^2-\Delta^2}$ and $\overline{E} = -\frac{\omega-\Omega}{2}$:

$$\boxed{H = \Omega A^\dagger A + \overline{E}}.$$

If we consider $\omega^2 < \Delta^2$, the coefficient Ω of the Hamiltonian becomes imaginary and the operator stops being Hermitian. This leads to an unphysical Quantum system, and it is not stable.

Q5 Fourier Transform.

Questions:

a) Show that

$$G(\vec{x}\,', \vec{x}) = \left\langle \vec{x}\,' \left| \frac{1}{E - \frac{-\nabla^2}{2m_R} \pm i\delta} \right| \vec{x} \right\rangle,$$
$$= -2m_R \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{k^2 - q^2 \mp i\delta} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')},$$
$$= -2m_R \frac{1}{4\pi R} e^{\pm iqR},$$

where $R = |\vec{x} - \vec{x}'|$ and $E = \frac{q^2}{2m_R}$.

b) Show that $\frac{e^{-mr}}{r}$ and $\frac{4\pi}{k^2+m^2}$ is a pair of (3D) Fourier transform.

Solutions

a) First, we have to understand the notation. Let L be an operator acting on some Hilbert space. Let G be the Green operator defined by

$$LG = 1,$$

where 1 is the identity operator. This means that for any state $|f\rangle$, $LG\,|f\rangle = |f\rangle$. Now, let's project into the position basis:

$$\langle x'|LG|f\rangle = \langle x'|f\rangle = f(x').$$

We can use resolution of the identity $\int |x\rangle \langle x|\,\mathrm{d}x = 1$ and insert it in the middle:

$$\langle x'|LG|f\rangle = \int \mathrm{d}x\, \langle x'|LG|x\rangle \langle x|f\rangle\,,$$
$$= \int \mathrm{d}x\, \langle x'|LG|x\rangle\, f(x) = f(x').$$

This implies that $\langle x'|LG|x\rangle = \delta(x' - x)$. As this is the property that defines the Dirac Delta distribution. Let's define:

$$G(x, x') \equiv \langle x|G|x'\rangle\,,$$

and then consider

$$\langle x'|LG|x\rangle = \int \mathrm{d}y\, \langle x'|L|y\rangle \langle y|G|x\rangle\,,$$
$$= \int \mathrm{d}y\, \langle x'|L|y\rangle\, G(y, x).$$

The matrix element of L is $\langle x'|L|y\rangle = L_{x'}\delta(x'-y)$:

$$\begin{aligned}\langle x'|LG|x\rangle &= \int \mathrm{d}y\, L_{x'}\delta(x'-y)G(y,x),\\ &= L_{x'}G(x',x).\end{aligned}$$

Therefore, $L_x G(x',x) = \delta(x'-x)$. Given that G is the "inverse" of L, in the operator sense, we can also write

$$\langle x'|G|x\rangle = \left\langle x'\middle|\frac{1}{L}\middle|x\right\rangle.$$

This is still valid in 3-dimensions:

$$G(\vec{x}\,',\vec{x}) = \langle \vec{x}\,'|G|\vec{x}\rangle = \left\langle \vec{x}\,'\middle|\frac{1}{L}\middle|\vec{x}\right\rangle.$$

And we now have a concise picture of what

$$\left\langle \vec{x}\,'\middle|\frac{1}{E-\frac{-\nabla^2}{2m_R}\pm i\delta}\middle|\vec{x}\right\rangle$$

refers to.

Now, define the 3-dimensional Fourier transform of $G(\vec{x}\,',\vec{x})$ as

$$\tilde{G}(k) = \int \mathrm{d}^3x\; e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}G(\vec{x}\,',\vec{x}).$$

Transforming it back:

$$G(\vec{x}\,',\vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\tilde{G}(k).$$

If we substitute this inverse Fourier transform version all the way back to the start, the derived properties would be intact. But we would have the inverse of the effect of L_x on the exponential $e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$. For example, if the operator was ∇^2, we would have $\frac{1}{(i)^2k^2}$, as $\nabla^2 e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = (ik)^2$. Therefore $\tilde{G}(k)$ is exactly $\frac{1}{-k^2}$ in this case. The notation of having the operator in the denominator helps us compute these quantities in a more straightforward way. Without having to rely on inverse of operator which involve integrations to find if we are in the position basis.

Now to solve the actual problem, let $L = E + \frac{\nabla^2}{2m_R} \pm i\delta$:

$$\begin{aligned}G(\vec{x}\,,\vec{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{E-\frac{k^2}{2m_r}\pm i\delta},\\ &= -2m_R\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{-2m_RE+k^2\mp i\delta},\\ &= -2m_R\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{k^2-q^2\mp i\delta}.\end{aligned}$$

Where we have defined used $E = \frac{q^2}{2m_R}$ and used the fact that δ tends to zero, therefore it doesn't care about any additional constants. To solve this integral, we can change into spherical coordinates $\mathrm{d}^3k = k^2\,\mathrm{d}\Omega\,\mathrm{d}k$, where $\mathrm{d}\Omega = \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi$ and we align the k_z-axis with the vector $\vec{R} = \vec{x} - \vec{x}'$ such that the following dot product is true:

$$\vec{k} \cdot \vec{R} = |\vec{k}||\vec{R}|\cos(\theta) = kR\cos(\theta).$$

We continue with the computation:

$$\begin{aligned}
G(R) &= -\frac{2m_R}{(2\pi)^3}\int_0^\infty \mathrm{d}k\,\frac{k^2}{k^2 - q^2 \mp i\delta}\int_0^\pi \mathrm{d}\theta\,\sin(\theta)e^{ikR\cos(\theta)}\int_0^{2\pi}\mathrm{d}\phi\,,\\
&= -\frac{2m_R}{(2\pi)^2}\int_0^\infty \mathrm{d}k\,\frac{k^2}{k^2 - q^2 \mp i\delta}\int_0^\pi \mathrm{d}\theta\,\sin(\theta)e^{ikR\cos(\theta)},\\
&= -\frac{2m_R}{(2\pi)^2}\int_0^\infty \mathrm{d}k\,\frac{k^2}{k^2 - q^2 \mp i\delta}\int_1^{-1}(-1)e^{ikRx}\,\mathrm{d}x\,,\\
&= -\frac{2m_R}{(2\pi)^2}\int_0^\infty \mathrm{d}k\,\frac{k^2}{k^2 - q^2 \mp i\delta}\frac{2\sin(kR)}{kR},\\
&= -\frac{2m_R}{2\pi^2 R}\int_0^\infty \mathrm{d}k\,\frac{k\sin(kR)}{k^2 - q^2 \mp i\delta},\\
&= -\frac{2m_R}{4\pi^2 R}\int_{-\infty}^\infty \mathrm{d}k\,\frac{k\sin(kR)}{k^2 - q^2 \mp i\delta}.
\end{aligned}$$

In the last line, we have used the fact that the whole function is even. Because k^2 is even, k is odd and $\sin(k)$ are odd, so their product is even. Then we can extend the integral to cover the whole real line for k. Now, we can use $\mp i\delta$ to apply the residue theorem. As we have moved the poles away from the real line, into the complex plane using that quantity. But we have a problem, to select a contour we should be able to choose one of the half-planes; where the function vanishes at the bigger the contour is on that plane. But $\sin(kR)k$ blows up at either plane. Given that $\sin(kR)k = k\frac{e^{ikR} - e^{-ikR}}{2i}$, and $R \in \mathbb{R}^+$, each tends to infinity on each plane. Therefore, we must analyze each exponential separately. Let

$$J(R) = \int_{-\infty}^\infty \frac{ke^{ikR}}{k^2 - q^2 \mp i\delta}.$$

Notice that now everything inside the integral except the exponential is odd. If we let $R \to -R$ we get:

$$J(-R) = \int_{-\infty}^\infty \frac{ke^{-ikR}}{k^2 - q^2 \mp i\delta},$$

now let $k \to -k$,

$$J(-R) = -\int_{-\infty}^\infty \frac{ke^{ikR}}{k^2 - q^2 \mp i\delta} = -J(R).$$

Now back to the main integral:

$$\begin{aligned}
G(R) &= -\frac{2m_R}{4\pi^2 R}\int_{-\infty}^{\infty} \mathrm{d}k\, \frac{k\sin(kR)}{k^2-q^2\mp i\delta}, \\
&= -\frac{2m_R}{4\pi^2 R}\frac{1}{2i}[J(R)-J(-R)], \\
&= -\frac{2m_R}{4\pi^2 R}\frac{1}{i}J(R), \\
&= -\frac{2m_R}{4\pi^2 Ri}\int_{-\infty}^{\infty} \frac{ke^{ikR}}{k^2-q^2\mp i\delta}.
\end{aligned}$$

Now we can select a plane to close the contour. Let $k = x + iy$:

$$e^{ikR} = e^{i(x+iy)R} = e^{ixR}e^{-yR}$$

at positive y (upper plane), the exponential decays and tends to zero. At negative y (lower plane), the exponential tends to infinity. Therefore we musts close at k with positive imaginary part. Now onto analyze the poles. We have two cases:

- $k^2 - q^2 - i\delta$:

$$k^2 = q^2 + i\delta \to k_{1,2} = \pm\sqrt{q^2+i\delta}.$$

  We see that $k_1 \approx q + i\delta$ and $k_2 \approx -q - i\delta$, given that δ tends to zero. So, k_1 is in the upper plane and k_2 in the lower. We must compute the Residue with k_1. Also notice that $k_1 - k_2 = 2k_1 = 2q$ in the limit.

$$\begin{aligned}
G(R) &= -\frac{2m_R}{4\pi^2 Ri}(2\pi i)\,\mathrm{Res}_{k=k_1}\left[\frac{ke^{ikR}}{(k-k_1)(k-k_2)}\right], \\
&= -\frac{2m_R}{4\pi^2 Ri}(2\pi i)\lim_{k\to k_1}(k-k_1)\left(\frac{ke^{ikR}}{(k-k_1)(k-k_2)}\right), \\
&= -\frac{2m_R}{4\pi^2 Ri}(2\pi i)\lim_{k\to k_1}\left(\frac{ke^{ikR}}{(k-k_2)}\right), \\
&= -\frac{2m_R}{4\pi^2 Ri}(2\pi i)\left(\frac{k_1e^{ik_1R}}{(k_1-k_2)}\right), \\
&= -\frac{2m_R}{4\pi^2 Ri}(2\pi i)\left(\frac{qe^{iqR}}{2q}\right), \\
&= -\frac{2m_R}{4\pi R}e^{iqR}.
\end{aligned}$$

- $k^2 - q^2 + i\delta$:

$$k^2 = q^2 - i\delta \to k_{1,2} = \pm\sqrt{q^2-i\delta}.$$

  We see that $k_1 \approx q - i\delta$ and $k_2 \approx -q + i\delta$, given that δ tends to zero. So, k_2 is in the upper plane and k_1 in the lower. We must compute the Residue with k_2. Also notice that $k_2 - k_1 = 2k_2 = -2q$ in the limit.

$$
\begin{aligned}
G(R) &= -\frac{2m_R}{4\pi^2 Ri}(2\pi i)\,\mathrm{Res}_{k=k_2}\left[\frac{ke^{ikR}}{(k-k_1)(k-k_2)}\right], \\
&= -\frac{2m_R}{4\pi^2 Ri}(2\pi i)\lim_{k\to k_2}(k-k_2)\left(\frac{ke^{ikR}}{(k-k_1)(k-k_2)}\right), \\
&= -\frac{2m_R}{4\pi^2 Ri}(2\pi i)\lim_{k\to k_2}\left(\frac{ke^{ikR}}{(k-k_1)}\right), \\
&= -\frac{2m_R}{4\pi^2 Ri}(2\pi i)\left(\frac{k_2e^{ik_2R}}{(k_2-k_1)}\right), \\
&= -\frac{2m_R}{4\pi^2 Ri}(2\pi i)\left(\frac{-qe^{-iqR}}{-2q}\right), \\
&= -\frac{2m_R}{4\pi R}e^{-iqR}.
\end{aligned}
$$

Finally, we can group both results to finish the question:

$$
\boxed{G(\vec{x}\,', \vec{x}) = \frac{-2m_R}{4\pi R}e^{\pm iqR}}.
$$

b) To prove that these functions are a pair of (3D) Fourier transform, we must show that Fourier transforming one gives the other one and vice-versa.

- FT $\frac{4\pi}{k^2+m^2}$:

$$
\begin{aligned}
\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{4\pi e^{i\vec{k}\cdot\vec{r}}}{k^2+m^2} &= \frac{1}{(2\pi)^3}\int_0^\infty \mathrm{d}k\,\frac{4\pi k^2}{k^2+m^2}\int_0^\pi \mathrm{d}\theta\sin(\theta)e^{ikr\cos(\theta)}\int_0^{2\pi}\mathrm{d}\phi\,, \\
&= \frac{2\pi}{2\pi^2}\int_0^\infty \mathrm{d}k\,\frac{k^2}{k^2+m^2}\int_{-1}^1 \mathrm{d}x\,e^{ikrx}, \\
&= \frac{1}{\pi}\int_0^\infty \frac{k^2}{k^2+m^2}\frac{2\sin(kr)}{kr}, \\
&= \frac{2}{\pi r}\int_0^\infty \frac{k\sin(kr)}{k^2+m^2}, \\
&= \frac{1}{\pi r}\int_{-\infty}^\infty \frac{k\sin(kr)}{k^2+m^2}.
\end{aligned}
$$

We can see that this integral hast the same structure as before. The only difference is that the poles are in the imaginary line, $k^2+m^2 = (k-im)(k+im)$. Therefore, we can apply the same strategy: divide $\sin(kr)$ into two integrals and decide on which half of the complex plane we want to close the contour integral. Let

$$
J(r) = \int_{-\infty}^\infty \frac{ke^{ikr}}{k^2+m^2}.
$$

Everything inside the integral except the exponential is odd. Let $r \to -r$:

$$J(-r) = \int_{-\infty}^{\infty} \frac{ke^{-ikr}}{k^2+m^2},$$

and let $k \to -k$:

$$J(-r) = -\int_{-\infty}^{\infty} \frac{ke^{ikr}}{k^2+m^2} = -J(r).$$

Back to the original integral:

$$\begin{aligned}
\frac{1}{\pi r}\int_{-\infty}^{\infty} \frac{k\sin(kr)}{k^2+m^2} &= \frac{1}{\pi}\frac{1}{2i}[J(r) - J(-r)], \\
&= \frac{1}{2\pi i r}[J(r) + J(r)], \\
&= \frac{1}{\pi i r} J(r), \\
&= \frac{1}{\pi i r}\int_{-\infty}^{\infty} \frac{ke^{ikr}}{k^2+m^2}.
\end{aligned}$$

As before, the integral will only converge when we close the contour in the upper half plane ($y > 0$):

$$e^{ikr} = e^{i(x+iy)r} = e^{ixr}e^{-yr}.$$

Therefore, we pick the pole $k = im$:

$$\begin{aligned}
\frac{1}{\pi i r}\int_{-\infty}^{\infty} \frac{ke^{ikr}}{k^2+m^2} &= \frac{1}{\pi i r} 2\pi i \operatorname{Res}_{k=im}\left[\frac{ke^{ikr}}{k^2+m^2}\right], \\
&= \frac{1}{\pi i r} 2\pi i \lim_{k=im} (k - im)\left(\frac{ke^{ikr}}{(k-im)(k+im)}\right), \\
&= \frac{2}{r}\lim_{k=im}\left(\frac{ke^{ikr}}{k+im}\right), \\
&= \frac{2}{r}\left(\frac{(im)e^{i(im)r}}{2im}\right), \\
&= \frac{e^{-mr}}{r}.
\end{aligned}$$

Therefore:

$$\boxed{\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{4\pi e^{i\vec{k}\cdot\vec{r}}}{k^2+m^2} = \frac{e^{-mr}}{r}}.$$

- IFT $\frac{e^{-mr}}{r}$:

$$\begin{aligned}
\int \mathrm{d}^3 r \frac{e^{-mr}}{r} e^{-i\vec{k}\cdot\vec{r}} &= \int_0^\infty \mathrm{d}r\, \frac{r^2 e^{-mr}}{r} \int_0^\pi \mathrm{d}\theta \sin(\theta) e^{-ikr\cos(\theta)} \int_0^{2\pi} \mathrm{d}\phi\,, \\
&= 2\pi \int_0^\infty \mathrm{d}r\, r e^{-mr} \int_{-1}^{1} \mathrm{d}x\, e^{-ikx}, \\
&= 2\pi \int_0^\infty \mathrm{d}r\, r e^{-mr} \frac{2\sin(kr)}{kr}, \\
&= \frac{4\pi}{k} \int_0^\infty \mathrm{d}r\, e^{-mr} \sin(kr).
\end{aligned}$$

To finish this integral, let us proceed by integration by parts: $\int u\, \mathrm{d}v = uv - \int v\, \mathrm{d}u$. Let $u = \sin(kr)$ and $\mathrm{d}v = e^{-mr}$. Therefore, $\mathrm{d}u = k\cos(kr)\,\mathrm{d}r$ and $v = \frac{e^{-mr}}{m}$.

$$\begin{aligned}
\frac{4\pi}{k} \int_0^\infty \mathrm{d}r\, e^{-mr} \sin(kr) &= \frac{4\pi}{k} \frac{e^{-mr}\sin(kr)}{m}\bigg|_0^\infty - \frac{4\pi}{m} \int_0^\infty e^{-mr}\cos(kr)\,\mathrm{d}r\,, \\
&= -\frac{4\pi}{m} \left[\frac{\cos(kr) e^{-mr}}{m}\bigg|_0^\infty + \int_0^\infty \frac{k e^{-mr}\sin(kr)}{m}\,\mathrm{d}r \right], \\
&= -\frac{4\pi}{m} \left[-\frac{1}{m} + \frac{k}{m} \int_0^\infty e^{-mr}\sin(kr)\,\mathrm{d}r \right], \\
&= \frac{4\pi}{m^2} - \frac{4\pi k}{m^2} \int_0^\infty e^{-mr}\sin(kr)\,\mathrm{d}r\,.
\end{aligned}$$

Notice that we have the same integral on both sides, which we can move to the left-hand side and factor out the constants:

$$\begin{aligned}
\frac{4\pi}{k}\left(1 + \frac{k^2}{m^2}\right) \int_0^\infty \mathrm{d}r\, e^{-mr}\sin(kr) &= \frac{4\pi}{m^2}, \\
\frac{4\pi}{k}\left(\frac{m^2 + k^2}{m^2}\right) \int_0^\infty \mathrm{d}r\, e^{-mr}\sin(kr) &= \frac{4\pi}{m^2}, \\
\frac{4\pi}{k} \int_0^\infty \mathrm{d}r\, e^{-mr}\sin(kr) &= \frac{4\pi}{m^2} \frac{m^2}{k^2 + m^2}.
\end{aligned}$$

Therefore:

$$\boxed{\int \mathrm{d}^3 x e^{-mr} e^{-i\vec{k}\cdot r} = \frac{4\pi}{k^2 + m^2}}.$$

Proving the claim.

Q6 Gauge Transformation

Questions:

Consider the QED Lagrangian:

$$\mathcal{L}(x) = \overline{\psi}(x)(i\gamma^\mu \partial_\mu - m - e\gamma^\mu A_\mu(x))\psi(x) - \frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x).$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

a) Derive the equations of motion for ψ and A_μ fields.

b) Under a gauge transformation

$$\psi(x) \to e^{-i\alpha(x)}\psi(x)$$

with $\alpha(x)$ being a scalar field, how should $A_\mu(x)$ transform in order to preserve the Lagrangian? I.e.

$$\mathcal{L}' \to \mathcal{L}.$$

c) work out the Noether's current for the gauge transformation and show that it is conserved.

Solutions

a) To derive the equations of motion, we make use of the Euler-Lagrange equation for each field:

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \overline{\psi})} \right) = 0, \qquad \frac{\partial \mathcal{L}}{\partial A_\alpha} - \partial_\beta \left(\frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} \right) = 0.$$

Noting that we have to use $\overline{\psi}(x)$ in the EL-equation to obtain the Equation of motion for ψ.

We compute each term:

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = (i\gamma^\mu \partial_\mu - m - e\gamma^\mu A_\mu(x))\psi(x).$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \overline{\psi})} = 0.$$

No derivatives of $\overline{\psi}(x)$ present in the Lagrangian. Therefore the EL-equation reads:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial \overline{\psi}} - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \overline{\psi})} \right), \\ &= (i\gamma^\mu \partial_\mu - mc)\psi(x) - e\gamma^\mu A_\mu(x)\psi(x). \end{aligned}$$

$$\therefore \boxed{(i\gamma^\mu \partial_\mu - mc)\psi(x) = e\gamma^\mu A_\mu(x)\psi(x)}.$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\alpha} &= -e\frac{\partial}{\partial A_\alpha}\left(\overline{\psi}(x)\gamma^\mu A_\mu(x)\psi(x)\right), \\
&= -e\overline{\psi}(x)\gamma^\mu \delta^\alpha_\mu \psi(x), \\
&= -e\overline{\psi}(x)\gamma^\alpha \psi(x).
\end{aligned}$$

For the last derivative, first compute:

$$\begin{aligned}
F^{\mu\nu}(x)F_{\mu\nu}(x) &= (\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x))(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)), \\
&= \partial_\mu A_\nu(x)\partial^\mu A^\nu(x) - \partial_\mu A_\nu(x)\partial^\nu A^\mu(x) \\
&- \partial_\nu A_\mu(x)\partial^\mu A^\nu(x) + \partial_\nu A_\mu(x)\partial^\nu A^\mu(x), \\
&= 2(\partial_\mu A_\nu(x)\partial^\mu A^\nu(x) - \partial_\mu A_\nu(x)\partial^\nu A^\mu(x)).
\end{aligned}$$

Where we have renamed the summed indices in the last two terms of line 2. And now:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} &= -\frac{1}{4}\frac{\partial}{\partial(\partial_\beta A_\alpha)}\left(F_{\mu\nu}(x)F^{\mu\nu}(x)\right), \\
&= -\frac{1}{2}\frac{\partial}{\partial(\partial_\beta A_\alpha)}(\partial_\mu A_\nu(x)\partial^\mu A^\nu(x) - \partial_\mu A_\nu(x)\partial^\nu A^\mu(x)), \\
&= -\frac{1}{2}[\delta^\beta_\mu \delta^\alpha_\nu \partial^\mu A^\nu(x) + \partial_\mu A_\nu(x)\eta^{\beta\mu}\eta^{\alpha\nu} - \delta^\beta_\mu \delta^\alpha_\nu \partial^\nu A^\mu(x) - \partial_\mu A_\nu(x)\eta^{\beta\nu}\eta^{\alpha\mu}], \\
&= -\frac{1}{2}[\partial^\beta A^\alpha(x) + \partial^\beta A^\alpha(x) - \partial^\alpha A^\beta(x) - \partial^\alpha A^\beta(x)], \\
&= -[\partial^\beta A^\alpha(x) - \partial^\alpha A^\beta(x)], \\
&= -F^{\beta\alpha}(x).
\end{aligned}$$

Therefore, the EL-equation reads:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial A_\alpha} - \partial_\beta\left(\frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)}\right), \\
&= -e\overline{\psi}(x)\gamma^\mu\psi(x) + \partial_\beta F^{\beta\alpha}(x).
\end{aligned}$$

$$\therefore \boxed{\partial_\beta F^{\beta\alpha}(x) = e\overline{\psi}(x)\gamma^\alpha\psi(x)}.$$

b) First, expand the parenthesis in the Lagrangian to see more explicitly each term.

$$\mathcal{L} = i\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - m\overline{\psi}(x)\psi(x) - e\overline{\psi}(x)\gamma^\mu A_\mu(x)\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x).$$

Apply the transformation to each term in the Lagrangian. With

$$\partial_\mu(e^{-i\alpha(x)}\psi(x)) = \left[-i\partial_\mu\alpha(x)\psi(x) + \partial_\mu\psi(x)\right]e^{-i\alpha(x)}.$$

We automatically notice that all terms except the one that contains the derivative of $\psi(x)$ will remain the same, as $e^{\pm i\alpha(x)}$ is just a number and commutes with any matrix quantity. Therefore, we just have to analyze that derivative term along with the coupling term to determine how $A_\mu(x)$ should transform.

$$\begin{aligned}
\mathcal{L}'_{Kin} &= ie^{+i\alpha(x)}\overline{\psi}(x)\gamma^\mu\partial_\mu(e^{-i\alpha(x)}\psi(x)) - ee^{+i\alpha(x)}\overline{\psi}(x)\gamma^\mu A'_\mu(x)\psi(x)e^{-i\alpha(x)}, \\
&= ie^{+i\alpha(x)}\overline{\psi}(x)\gamma^\mu\left[-i\partial_\mu\alpha(x)\psi(x) + \partial_\mu\psi(x)\right]e^{-i\alpha(x)} - e\overline{\psi}(x)\gamma^\mu A'_\mu(x)\psi(x) \\
&= \overline{\psi}(x)\partial_\mu\alpha(x)\psi(x) + i\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - e\overline{\psi}(x)\gamma^\mu A'_\mu(x)\psi(x), \\
&= i\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - i\overline{\psi}(x)\gamma^\mu\left[A'_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)\right]\psi(x), \\
&\overset{!}{=} \mathcal{L}_{Kin} = i\overline{\psi}(x)\gamma^\mu\partial_\mu(x) - i\overline{\psi}(x)\gamma^\mu A_\mu(x)\psi(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_\mu(x) &= A'_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x), \\
\therefore A'_\mu(x) &= A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x).
\end{aligned}$$

To ensure this is consistent, we check that the field-strength term is already gauge-invariant. Consider

$$\partial_\mu(A'_\nu(x)) = \partial_\mu A_\nu(x) + \frac{1}{e}\partial_\mu\partial_\nu\alpha(x).$$

$$\begin{aligned}
F'_{\mu\nu}(x)F'^{\mu\nu}(x) &= (\partial_\mu A'_\nu(x) - \partial_\nu A'_\mu(x))(\partial^\mu A'^\nu(x) - \partial^\nu A'^\mu(x)), \\
&= \left(\partial_\mu A_\nu(x) + \frac{1}{e}\partial_\mu\partial_\nu\alpha(x) - \partial_\nu A_\mu(x) - \frac{1}{e}\partial_\nu\partial_\mu\alpha(x)\right)\times \\
&\times\left(\partial^\mu A^\nu(x) + \frac{1}{e}\partial^\mu\partial^\nu\alpha(x) - \partial^\nu A^\mu(x) - \frac{1}{e}\partial^\nu\partial^\mu\alpha(x)\right), \\
&= (\partial_\mu A_\nu(x) - \partial^\nu A^\mu(x))(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)), \\
&= F_{\mu\nu}(x)F^{\mu\nu}(x).
\end{aligned}$$

Mixed derivatives commute ($\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$). So we see that indeed, the transformation is correct and $A_\mu(x)$ should transform as:

$$\boxed{A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)}.$$

c) To compute the Noether Current for a gauge transformation, we can use the Global Transformation version (Aldrovandi & Pereira, 2019). But keeping in mind that although this is a conserved current, it is not an observable because of gauge redundancy:

$$J^\mu_\alpha = -\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\frac{\overline{\delta}\phi_i}{\delta\omega^\alpha} + \mathcal{L}\frac{\delta x^\mu}{\delta\omega^\alpha}\right].$$

Here, we just have one generator of the $U(1)$ group, therefore we have no α indices. And this is an internal symmetry, therefore $\frac{\delta x^\mu}{\delta \omega^\alpha} = 0$. The form functional form (Aldrovandi & Pereira, 2019) is defined as

$$\overline{\delta}\phi_i(x) = \phi_i'(x) - \phi_i(x).$$

Which reflects a change in the fields at the same point in spacetime. Here $\psi_i'(x) = e^{-i\alpha(x)}\psi(x)$ and $\overline{\psi}'(x) = e^{+i\alpha(x)}\psi(x)$.

$$\begin{aligned}
\overline{\delta}\psi(x) &= (e^{-i\alpha(x)} - 1)\psi(x) \approx -i\alpha(x)\psi(x), \\
\overline{\delta\psi}(x) &= (e^{+i\alpha(x)} - 1)\psi(x) \approx +i\alpha(x)\psi(x).
\end{aligned}$$

Which leads to:

$$\begin{aligned}
\frac{\overline{\delta}\psi(x)}{\delta\alpha} &= -i\psi(x) \\
\frac{\overline{\delta\psi}(x)}{\delta\alpha} &= +i\overline{\psi}(x).
\end{aligned}$$

We have to sum over all fields ϕ_i in the Lagrangian. But the Lagrangian only depends on the derivatives of $\psi(x)$. This simplifies the current to:

$$\begin{aligned}
J^\mu &= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\overline{\delta}\psi(x), \\
&= -(i\overline{\psi}(x)\gamma^\mu)(-i\psi(x)).
\end{aligned}$$

Therefore

$$\boxed{J^\mu = -\overline{\psi}(x)\gamma^\mu\psi(x)}.$$

References

Aldrovandi, R., & Pereira, J. G. (2019). *An elementary introduction to classical fields.* São Paulo, Brazil: Instituto de Física Teórica, UNESP. (Lecture notes)