Hadrons Fall2023

HW3 due 29/12/2023

(+2 points for handing in on time)

Q1 SU(2) Current Algebra.

- a) Given the Lagrangian

$$L = \int d^3x \, \bar{q}(x) \left(i\partial_\mu \gamma^\mu - \hat{m} \right) q(x)$$

where $q = \begin{pmatrix} u \\ d \end{pmatrix}$ and $\hat{m} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$; and the vector (V) and axial vector (AV) transformations

$$q \to e^{-i\alpha_V^a \, \tau^a} q$$
$$q \to e^{-i\gamma_5 \, \alpha_{AV}^a \, \tau^a} q$$

where $\tau^a = \frac{1}{2}\sigma^a$ are the generators of SU(2), $\sigma^{a=1,2,3}$ are the Pauli matrices. Derive an expression for the conserved charges Q_V^a and Q_{AV}^a. Take $x^0 = 0$ for simplicity.

- b) Construct the Hamiltonian H, and compute the commutators

$$[Q_V^a, H]$$
$$[Q_{AV}^a, H].$$

Recall the anti-commutation relations among field operators (equal time), e.g.,

$$\{q^a(x), q^b(y)^\dagger\} = \delta^{(3)}(\vec{x} - \vec{y}) \, \delta^{ab},$$

and vanishes for fields of same type.

State the conditions on $\hat{m}$ for Q_V^a to be conserved. What about Q_A^a?

- c) Show that

$$[Q_{AV}^a, [Q_{AV}^a, H]] = \int d^3x \, \bar{q}(x) \, \{\tau^a, \{\tau^a, \hat{m}\}\} \, q(x).$$

This is the key step to proving the Gell-mann-Oakes-Renner relation:

$$m_\pi^2 = -\frac{1}{f_\pi^2}\langle 0|[Q_{AV}^a,[Q_{AV}^a,H]]|0\rangle$$
$$\text{for}\,\pi^0 \to -\frac{1}{f_\pi^2}\left(m_u\langle\bar{u}u\rangle + m_d\langle\bar{d}d\rangle\right).$$

Take $m_u = m_d = 5$ MeV, plug in the physical values of m_π and f_π and obtain the value of the chiral condensate $\langle\bar{u}u\rangle = \langle\bar{d}d\rangle$.

- d) Show that the charges satisfy

$$[Q_V^a, Q_V^b] = i\epsilon^{abc}Q_V^c$$
$$[Q_{AV}^a, Q_{AV}^b] = i\epsilon^{abc}Q_V^c,$$

and work out $[Q_V^a, Q_{AV}^b]$.

Verify that the Left-Right charges

$$Q_L^a = \frac{1}{2}(Q_V + Q_{AV})$$
$$Q_R^a = \frac{1}{2}(Q_V - Q_{AV})$$

are decoupled, i.e.

$$[Q_L^a, Q_L^b] = i\epsilon^{abc}Q_L^c$$
$$[Q_R^a, Q_R^b] = i\epsilon^{abc}Q_R^c$$
$$[Q_L^a, Q_R^b] = 0.$$

Q2 Self-energy (part II).

- a) Prove the Feynman parametrization:

$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{(xA+(1-x)B)^2}.$$

- b) The self energy Σ_R of a resonance is obtained from

$$\Sigma_R = ig^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{q_1^2-m_1^2+i\delta}\,\frac{1}{q_2^2-m_2^2+i\delta}$$

where

$$q_1 = q$$
$$q_2 = P - q$$
$$P^2 = s = (P^0)^2 - \vec{P}^2$$

are Minkowski 4-vectors. Use the Feynman parametrization to show that

$$\Sigma_R = ig^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{1}{((\tilde{q})^2 - \Delta)^2}$$

where

$$\tilde{q} = q - (1-x)P$$
$$\Delta = xm_1^2 + (1-x)m_2^2 - x(1-x)s - i\delta.$$

- c) After a shift of integration variable and a Wick's rotation:

$$d^4q \to id^4q_E$$
$$q^2 \to -q_E^2 = -(q_4^2 + \vec{q}^2),$$

we obtain

$$\Sigma_R = -g^2 \int_0^1 dx \int \frac{d^4q_E}{(2\pi)^4} \frac{1}{(q_E^2 + \Delta)^2}$$

where q_E is in Euclidean space.

Use the Schwinger proper time regularization and perform the momentum integral. Show that

$$\Sigma_R = \frac{g^2}{16\pi^2} \int_0^1 dx \ln \Delta + C.$$

This is the starting point of Q2 in HW02.

Hint: Recall the Schwinger proper time regularization scheme

$$\mathcal{A}^{-1} = \int_0^\infty dt\, e^{-t\mathcal{A}}$$
$$\ln \mathcal{A} = -\int_0^\infty dt\, \frac{1}{t} \left(e^{-t\mathcal{A}} - e^{-t\mathcal{I}}\right).$$

The following relation is also useful:

$$\mathcal{A}^{-2} = \int_0^\infty dt\, t\, e^{-t\mathcal{A}}.$$

Q3 Schwinger Proper Time Regularization (part II)

- a) Recall the use of Schwinger proper time regularization (HW02, Q1) to calculate

$$\int_{-\infty}^\infty \frac{dp_4}{2\pi} \frac{1}{p_4^2 + \omega^2} = \frac{1}{2\sqrt{\omega^2}}.$$

Now consider the divergent integral

$$W[\omega;\Lambda] = \int_{-\infty}^{\infty} \frac{dp_4}{2\pi}\ \ln\left(p_4^2+\omega^2\right).$$

Use the Schwinger proper time technique to regulate the integral, i.e. replace the lower limit of $t \to \frac{1}{\Lambda^2}$, and show that

$$W[\omega;\Lambda] = -\int_{1/\Lambda^2}^{\infty} dt\,\frac{1}{t^{\frac{3}{2}}}\,\frac{1}{2\sqrt{\pi}}\,e^{-t\omega^2} + C.$$

C is an integration constant.

- b) Study the integral at large Λ. Show that

$$W[\omega;\Lambda] = -\frac{\Lambda}{\sqrt{\pi}} + \omega + \mathcal{O}(1/\Lambda).$$

- c) This suggests the definition of a physical W function:

$$W_{\text{phys.}}[\omega] = \lim_{\Lambda\to\infty}\left(W[\omega;\Lambda] - W[0;\Lambda]\right) = \omega.$$

Re-derive the previous result (again!) via

$$\int_{-\infty}^{\infty} \frac{dp_4}{2\pi}\,\frac{1}{p_4^2+\omega^2} = \frac{1}{2\omega}\frac{\partial}{\partial\omega}W_{\text{phys.}}[\omega].$$

Q4 Klein-Gordon field equation.

- a) Show that the time-ordered 2-point function (via a contour integration)

$$\begin{aligned}\langle 0|T\{\phi(x)\phi(y)\}|0\rangle &= \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\delta}\,e^{-ip\cdot(x-y)}\\ &= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E(p)}\,e^{-iE(p)|x^0-y^0|}\,e^{i\vec{p}\cdot(\vec{x}-\vec{y})}.\end{aligned}$$

where $E(p) = \sqrt{p^2+m^2}$.

- b) Show that

$$(\partial^2+m^2)\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = -i\delta^{(4)}(x-y).$$