# Hadrons Fall2023

## HW4 due 30/01/2024

(+2 points for handing in on time)

### Q1 SU(2) Transformation for Composite Operators.

- a) Given the vector (V) and axial vector (AV) transformations

$$q \to e^{-i\alpha_V^a \tau^a} q$$
$$q \to e^{-i\gamma_5 \alpha_{AV}^a \tau^a} q,$$

work out (to leading order in $\alpha$'s) the transformations of the operators

$$p1 = \bar{q}(x)q(x)$$

and

$$p2 = 2i\bar{q}(x)\gamma_5\tau^a q(x)$$

under V and AV.

- b) Construct the transformation rules for $\sigma$ and $\pi^a$ fields such that

$$\bar{q}(x)(\sigma - 2i\pi^a\gamma_5\tau^a)q(x)$$

is invariant under V and AV.

### Q2 Model of Resonances.

A simple model of the scattering matrix $S$ for a resonance exchange is given by

$$S = e^{2i\delta} = 1 + 2iqf(s)$$
$$f(s) = \frac{K(s)}{s - m_{\text{bare}}^2 - \Sigma(s)}.$$

where $\delta(s)$ is the scattering phase shift, $f$ is the scattering amplitude of a resonance exchange. The proportionality constant $K$ is irrelevant but can be worked out:

$$K = \frac{2\,\text{Im}\Sigma}{2\,q(s)}$$

where $q(s)$ is the COM momentum.

- a) Prove the identity:

$$e^{2i\delta} = 1 + 2\,i\sin(\delta)e^{i\delta}$$

.

- b) Write down an expression of the phase shift in terms of $\Sigma(s)$.
- c) The self energy of a P-wave resonance, after a (rather tedious) QFT calculation (with some additional manipulations), gives

$$\Sigma_R(s) = -\frac{g^2}{8\pi^2}\int_0^1 dx\,\Delta\,\ln\Delta$$
$$\Delta = xm_1^2 + (1-x)m_2^2 - x(1-x)s - i\delta.$$

  Show that the imaginary part reads

$$\mathrm{Im}\Sigma_R = -\frac{4}{3}\times\left(\frac{1}{2}\,g^2\,\frac{q^3}{4\pi\sqrt{s}}\,\theta(\sqrt{s}-m_1-m_2)\right).$$

  (Compare with Q2 in HW2, a $q(s)^2$ factor naturally emerges!)
- d) Adjust the two parameters: $m_{\mathrm{bare}}$ and $g^2$ such that the model qualitatively describes the $\Delta(1232)$ resonance. Plot the result against $\sqrt{s}$.

## Q3 Zeta Functions

- a) The Riemann Zeta function is defined as

$$\zeta_{\hat{R}}(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Show that

$$\frac{d}{ds}\zeta_{\hat{R}}(s)|_{s\to 0} = -\sum_{n=1}^{\infty}\ln n$$
$$= -\mathrm{tr}\ln\hat{\mathrm{R}},$$

where $\hat{R}$ is an operator with eigenvalues $n = 1, 2, 3, \ldots$.

This allows a generalization of the definition of Zeta function for an arbitrary operator $\hat{O}$ (with eigenvalues $\lambda_n$) via

$$\zeta_{\hat{O}}(s) = \sum_{\lambda_n}\frac{1}{\lambda_n^s}$$

and

$$\zeta'_{\hat{O}}(0) = -\ln\det\hat{O}.$$

- b) Consider (again! HW03 Q3) the integral

$$\mathcal{I} = \int_{-\infty}^{\infty} \frac{dp_4}{2\pi} \, \ln\left(p_4^2 + \omega^2\right) = \text{Div.} + \omega.$$

Use the Riemann summation to show that

$$\begin{aligned} \mathcal{I} &= \Delta p \times \ln \det \hat{D} \\ &= -\Delta p \times \zeta'_{\hat{D}}(0). \end{aligned}$$

where the operator $\hat{D}$ has eigenvalues

$$(2\pi n \Delta p)^2 + \omega^2$$

with $n$ being integers

$$n = -\infty, \ldots \infty.$$

- c) Show that

$$\begin{aligned} \zeta_{\hat{D}}(s) &= \frac{1}{\omega^{2s}} + 2 \sum_{m=1}^{\infty} \frac{1}{[(2\pi m \Delta p)^2 + \omega^2]^s} \\ &= \frac{1}{\omega^{2s}} + 2\,\zeta_{\hat{A}}(s). \end{aligned}$$

where the operator $\hat{A}$ has eigenvalues

$$(2\pi n \Delta p)^2 + \omega^2$$

with $n$ being integers

$$n = 1, 2, \ldots \infty.$$

- d) Derive the following expression:

$$\ln \det \hat{A} = \zeta_{\hat{R}}(0)\, \ln[4\pi^2 (\Delta p)^2] - 2\,\zeta'_{\hat{R}}(0) + \ln \frac{\sinh(\frac{\omega}{2\Delta p})}{\frac{\omega}{2\Delta p}}.$$

- e) Using the fact that

$$\begin{aligned} \zeta_{\hat{R}}(0) &= -\frac{1}{2} \\ \zeta'_{\hat{R}}(0) &= -\frac{1}{2} \ln(2\pi), \end{aligned}$$

verify (finally!) that

$$\mathcal{I} = -\Delta p \times \zeta'_{\hat{D}}(0)$$
$$= \omega.$$

Note that the divergent part derived previously disappears!

## Q4 A simple model of SCSB.

Consider a 4-fermion interaction model

$$Z = \int D\psi D\bar{\psi} e^{\int \bar{\psi}(i\gamma\cdot\partial - m)\psi + G\,(\bar{\psi}\psi)^2}.$$

Note that the integral is over an Euclidean space time.

- a) Prove the relation

$$e^{\int \frac{g^2}{2m_G^2}(\bar{\psi}\psi)^2} \propto \int D\sigma\, e^{\int \left(-g\sigma\bar{\psi}\psi - \frac{1}{2}m_G^2\sigma^2\right)}$$

and with this rewrite the 4-fermion interaction model as

$$Z \to \int D\sigma e^{\ln Z_F(M_F = m + g\sigma) - \int \frac{1}{2}m_G^2\sigma^2}.$$

where $Z_F(M_F)$ is the partition function for free fermions

$$\ln Z_F(M_F) = \mathrm{tr}\ln(i\gamma\cdot\partial - M_F)$$
$$= V_4\int^{\Lambda}\frac{d^3k}{(2\pi)^3}2\sqrt{k^2 + M_F^2} + \ldots$$

where $V_4 = \beta V$ is the Euclidean 4-volume. Identify $G$ in terms of $g$ and $m_G$. Notice the similarity with Fermi coupling constant for weak interaction.

- b) Suppose the functional integral over $\sigma$ is dominated by a certain $\bar{\sigma}$, such that

$$Z = \int D\sigma\, e^{\ln Z_F(M_F = m + g\sigma) - \int \frac{1}{2}m_G^2\sigma^2} \approx e^{-V_4\Gamma(\bar{\sigma})}.$$

Derive a condition for $\Gamma(\bar{\sigma})$ based on the steepest descent. This is called the gap equation.

- c) Solve the gap equation numerically. (You may set $g = 1$.) Plot $\sigma/\Lambda$ versus $G$ for $m = 0$ and $m \neq 0$. Derive an analytic expression for the critical coupling for the former case. What is the order of the phase transition?

- d) Find an explicit expression of the condensate via

$$\langle\bar{\psi}\psi\rangle = -\frac{\partial}{\partial m}\frac{\ln Z}{V_4}$$

Verify that the condensate is negative. In the model, one can compute it via

$$\begin{aligned} n_S = \langle\bar{\psi}\psi\rangle &= \frac{\partial}{\partial m}\Gamma(\langle\sigma\rangle, m) \\ &= \frac{\partial}{\partial m}\Gamma(\langle\sigma\rangle, m) + \left(\frac{\partial}{\partial\sigma}\Gamma(\sigma, m)\frac{\partial\sigma}{\partial m}\right)\Big|_{\sigma=\langle\sigma\rangle}. \end{aligned}$$

Explain why the second term will **not** contribute and relate the condensate to $\sigma$. (This is model dependent!)